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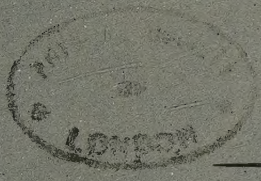

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ERRATA to VOL. VI. No. 6.

Page 250, line 4 from bottom, is is, read is.

„ 254, line 8 from bottom, for "As R′, is," „ "As R′ is,"

„ „ „ 4 „ bottom, for $\frac{\partial_s a_0(z)}{\partial z^s}$, read $\frac{\partial^s a_0(z)}{\partial z^s}$.

„ 308, „ 2 „ top, α-pimene, „ α-pinene.

ERRATUM to VOL. VI. No. 3.

Page 151, line 12, between "such that" and "$\omega(z)=a$,"
"$\omega(z)=\infty$," should be inserted.

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Studies on Camphor Series, III. Catalytic Action of Reduced Copper on dextro-Camphoroxime.

By

Shigeru Komatsu and Shozo Yamaguchi.

(Received December 19, 1922.)

When camphoroxime was reduced with hydrogen in the presence of reduced nickel, it yielded bornylamine.[1] The authors, therefore, expected from the analogy between the catalytic action of copper and of nickel on acetophenonoxime,[2] that camphoroxime by the catalytic action of reduced copper and hydrogen, would be transformed into bornylamine. Contrary to our anticipation, the main reaction was actually a molecular rearrangement, with reduction as a side reaction.

35 grm. dextro-camphoroxime were passed with pure hydrogen over reduced copper heated to 200°, which had been prepared from copper oxide by reduction at 180° with hydrogen, following Prof. P. Sabatier's direction. In the progress of the reaction, there was observed the formation of a solid substance accompanied by a liquid one in the receiver. At the beginning of the reaction, however, a liquid substance only was formed. On the apparatus being opened after the reaction had ended, the odour of ammonia was perceptible, indicating that the decomposition of the oxime into camphor and hydroxylamine had taken place.

1 Aloy and Brustier, Bull. soc. chim., (4), **9**, 734 (1911).
Mailhe, C.R., **140**, 1691; **141**, 113 (1905).
3 La Catalyse en Chimie Organique, 1920, 18.

17 grm. of a yellowish green oily substance were obtained, which was treated with dilute hydrochloric acid and then subjected to distillation after drying with anhydrous sodium sulphate. 8 grm. of the distillate, thus obtained, were fractionated five times under 753–756 mm.

Fraction	yield	density	Carbon	Hydrogen	Nitrogen
Below 85°	0·2 grm.	——	——	——	——
140–185°	0·3 ,,	——	——	——	——
185–210°	0·6 ,,	——	78·81	10·92	——
210–220°	0·6 ,,	liquid.	78·96	10·92	4·81
	0·9 ,,	solid.	78·12	10·80	2·01
220–230°	1·5 ,,	0·9276	78·52	10·48	5·30
230–240°	0·5 ,,	0·9463	75·33	10·64	5·24
240–250°	0·5 ,,	1·0052	73·14	9·99	3·53
250–263°	1·5 ,,		71·87	10·08	1·85

As seen in the foregoing table, the third and fourth fractions seem to contain camphor and some nitrogenous substance which is the main constituent of the fifth and sixth fractions, the last fraction being composed of a carboxylic acid.

In another experiment, 60 grm. of dextro-camphoroxime were passed with hydrogen over reduced copper heated to 200°, and 35 grm. of reaction product were obtained which was separated mechanically into liquid and solid substances. The yield was 20 grm. of the former and 10 grm. of the latter.

The solid substance was treated with dilute hydrochloric acid, and then recrystallized from the hot aqueous solution. The white scaly crystals separated from the solution, were found to melt at 126°. They were soluble in organic solvents and hot water and the yield was 2 grm. It gave, on analysis;

C=71·77; 71·95; H=10·79; 10·67; N=8·35 (by Kjeldahl's method) ($C_{10}H_{17}ON$, requires C=71·81; H=10·23; N=8·38) $[\alpha]_D^{17}=-4°\cdot6'$ (in 96% alcohol solution).

When the substance was treated with zinc and conc. hydrochloric acid, it yielded a primary amine, and when hydrolysed with alcoholic potash was transformed into a carboxylic acid which gave a silver salt from its ammonium salt solution with silver nitrate.

The results of analysis quite agree with those of *a*-campholenamide obtained from *a*-campholenitrile by hydrolysis,[1] from *d*-camphoroxime by treating with mineral acids or acetyl chloride,[2] and by heating the ammonium salt of *a*-campholenic acid.[3]

With the object of further confirming the identity of our substance with *a*-campholenamide, it was hydrolysed with alcoholic potash into *a*-campholenic acid which was separated from the reaction products by steam distillation, and extracted with ether. The etherial solution was dried with anhydrous sodium sulphate, the solvent expelled by distillation, and the residue was then fractionated under 764·7mm.

Fraction	248–255°	255–258°	258–263°
Colour	colourless	pale yellow	yellow
n_D^{25}	1·4605	1·4614	1·4600

The second fraction boiled at 143° under 15 mm., $d_4^{25}=0{\cdot}9963$, $[a]_D^{16}=+17°{\cdot}40'$, and gave on analysis, C=71·25; H=9.62 ($C_{10}H_{16}O_2$, requires C=71·39; H=9·59).

It gave an insoluble silver salt from its ammonium salt solution with silver nitrate. The physical properties and analytical results of the fraction agreed with those of *a*-campholenic acid prepared by other investigators from *a*-campholenitrile, *a*-campholenamide[4] and bromcamphor.[5]

The liquid reaction product was subjected to steam distillation and the oily distillate in the flask was extracted with ether, and the extract, after the solvent had been distilled off, was fractionated four times under 753 mm.

Fraction	**yield**	**Carbon**	**Hydrogen**	**Nitrogen**
Below 200°	trace	——	——	——
200–210°	0·5 grm.	——	——	——
210–220°	3·5 „	78·02	10·93	——
		78·22	10·68	2·00
220–225°	2·0 „			

1 E. Nageli, Ber. D. Chem. Ges., **17**, 805 (1885); F. Tiemann, Ibid, **29**, 3006 (1896); A. Behal, C.R., **120**, 1167 (1895).

2 F. Tiemann, Ber. D. Chem. Ges., **28**, 1083 (1895).

3 H. Goldschmidt and R. Zurrer, Ibid, **17**, 2071; J. Kachler and F. V. Spitzer, Ibid, **17**, 2400 (1884).

4 F. Tiemann, Ber. D. Chem. Ges., **28**, 2167 (1895).

5 Kachler and Spitz, Ibid., **17**, 2400 (1884); Monatsh. Chem., **3**, 216 (1882).

Fraction	yield	Carbon	Hydrogen	Nitrogen
225–230°	2·7 „	80·95	11·14	——
230–240°	1·5 „	78·02	10·67	4·17
		78·15	10·56	——
240–250°	1·5 „	75·60	10·40	——
		75·64	10·59	——
250–260°	1·5 „	74·34	10·19	——
		74·19	10·38	——

The fraction B.p. 210–220°, was a mixture of solid and liquid substances which were separated from each other by filtration. The solid substances was confirmed to be dextro-camphor by its analysis and by the determination of melting point, 118°, and of the optical rotatory power of its oxime, $[\alpha]_D^{20} = -40°\cdot 9'$.

The two fractions B.p. 210–220° (liquid part) and B.p. 230–240° combined together, and were digested with barium hydroxide in a flask with reflux condenser for three hours to get campholenitrile free from campholenic acid and its acid amide. Some crystals were observed to be deposited on the wall of the condenser, which melted at 90–110°, and were too small in quantity to allow of their chemical nature being confirmed. The alkaline solution was subjected to steam distillation, and the oily distillate in the receiver was extracted with ether, and dried with anhydrous sodium sulphate. The residue remaining after distilling off the solvent was all distilled off between 220–230°. The alkaline solution remaining in the distillation flask, was acidified with sulphuric acid and again subjected to steam distillation, and an oily distillate then obtained was found to consist of *α*-campholenic acid.

The oily substance B.p. 220–230°, regarded to consist of *α*-campholenitrile, combined with the fractions boiling at 220–225°; and 225–230°, and the mixture was fractionated under 8 mm.

Fraction.			
	B.p.	98–118°	2·5 grm.
	B.p.	118–120°	2·5 grm.

The fraction B.p. 98–118°, assumed to be a mixture of *d*-camphor and *α*-campholenitrile, was digested with alcoholic potash to convert the latter substance into *α*-campholenic acid which, after separating from camphor by steam distillation, was transformed into ammonium salt. The ammonium salt was converted into silver salt which gave C=43·03; H=5·80; Ag=39·47 ($C_{10}H_{15}O_2Ag$ requires C=43·65; H=5·50; Ag=39·22).

The fraction B.p. 118–120° under 8 mm., $d_4^{25}=0{\cdot}91966$, $n_D^{25}=1{\cdot}4592$; $[\alpha]_D^{15}=+23°{\cdot}0$ (in 95% alcohol); gave on analysis C=80·65; H=10·32; N=9·6 (by Duma's method). ($C_{10}H_{15}N$ requires C=80.47; H=10·13; N=9·39).

It yielded, on hydrolysis with alcoholic potash, *α*-campholenic acid and ammonia. There can, therefore, be no doubt that the above fraction consists of pure *α*-campholenitrile which was obtained by E. Nageli and A. Behal from camphoroxime by the action of acetyl chloride,[1] also from the oxime by treating with sulphuric acid or hydrochloric acid.[2]

It was anticipated that the fractions B.p. 240–250°, 250–260°, of the liquid reaction product would contain *α*-campholenic acid, and in order to isolate the substance, it was dissolved in absolute ether and dry ammonia gas was passed over it, and the ammonium salt precipitated out from the solution separated, washed with ether, dried and dissolved in water. For the sake of comparison, the ammonium salt, converted into silver salt with silver nitrate, was analysed.

The fraction B.p. 230–240°, obtained from the liquid reaction product and also the fraction with the same boiling point isolated from the mixed fractions B.p. 210–220°, and 230–240° after treatment wth barium hydroxide, were both confirmed to be a mixture of campholenic acid and campholenitrile, by transforming into silver campholenate after separating the latter substance from the former by means of ammonia.

The acid washing of the solid reaction products above mentioned, on alkalining with caustic soda solution yielded bornylamine, which was separated by steam distillation, extracted with ether, and then converted into the hydrochloride.

For confirmation, the specific rotatory power of the hydrochloride in aqueous solution was determined with the following result:

$$[\alpha]_D=-36°{\cdot}9'$$

It gave the platinum double salt with platinum chloride.

Of the reaction products from *d*-camphoroxime, reduced copper and hydrogen at 200°, we have actually isolated the following substances.

d-*α*-campholenamide (laevo rotatory), *α*-campholenic acid, *d*-camphor, *α*-campholenitrile, and a trace of bornylamine.

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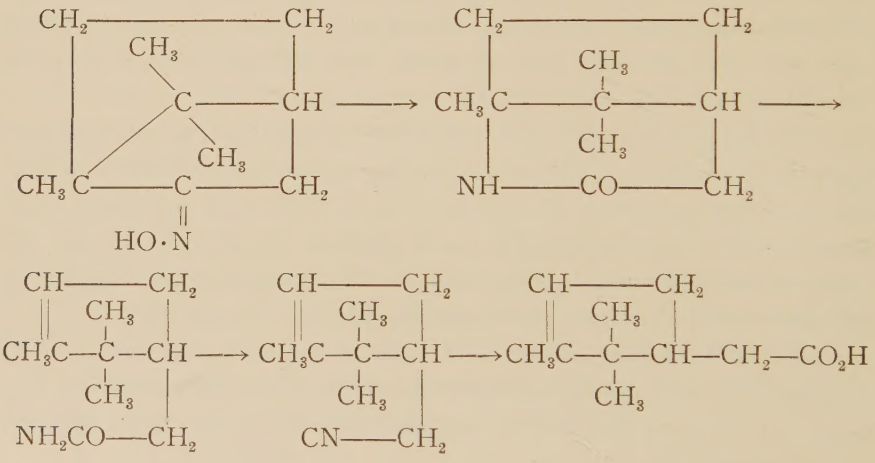
1 Loc. cit.

2 Loc. cit.

These compounds, except bornylamine and camphor, resulted from the molecular rearrangement of *d*-camphoroxime, which is effected by numerous reagents, such as acetyl chloride, sulphuric and hydrochloric acids.

The formation of these compounds, therefore, from comphoroxime by the catalytic action of reduced copper, should be explained by the same mechanism which applied in the reaction between camphoroxime and acetyl chloride, namely Beckmann's molecular rearrangement of the oxime (the reaction being probably analogous to that which occurs between nitrosocamphor and acetic acid); it occurs with rapture of the cyclopentamethylene ring, whereby the oxime converts into isoxime. The decomposition of the isoxime into campholenamide occurring soon after the rearrangement, is represented in the following fashions:



We are, now, forced to conclude that a large part of the oxime is is configurated in syn-form as indicated in the above formula, since it yields the acid amide by the rearrangement.

November 1922, Laboratory of Organic and Biochemistry.

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1 A. Nageli, Ber. D. Chem. Ges., **26**, 68 (1893); O. Manasse, Ibid., **26**, 242 (1893).

On Transcendental Integral and Transcendental Algebraic Functions and Algebraic Addition Theorems, II.

By

Ryô Yasuda.

(*Received December 20, 1922.*)

Chapter III.

ALGEBRAIC ADDITION THEOREMS FOR ANALYTIC FUNCTIONS OF MANY INDEPENDENT VARIABLES.

INTRODUCTION.

Definition. Let $f((z_i)) \equiv f(z_1, z_2, \ldots\ldots, z_n)$ be an analytic function of $z_1, z_2, \ldots\ldots, z_n$ defined all over the Gauss space. Let

$$f_0 \equiv f(z_{11}+z_{12},\ z_{21}+z_{22},\ \ldots\ldots\ldots\ldots,\ z_{n1}+z_{n2}),$$

$$f_{i1} \equiv f(z_{11}+z_{12},\ \ldots\ldots,\ z_{i-11}+z_{i-12},\ z_{i1},\ z_{i+11}+z_{i+12},\ \ldots\ldots,\ z_{n1}+z_{n2}),$$

$$f_{i2} \equiv f(z_{11}+z_{12},\ \ldots\ldots,\ z_{i-11}+z_{i-12},\ z_{i2},\ z_{i+11}+x_{i+12},\ \ldots\ldots,\ z_{n1}+z_{n2}),$$

$$(i=1,\ 2,\ \ldots\ldots,\ n).$$

When among these $(2n+1)$ functions, there exists a relation

$$(1) \qquad F(f_{11}, f_{12}, f_{21}, \ldots\ldots\ldots, f_{n2}, f_0)=0,$$

where F is an irreducible polynominal of $f_{11}, f_{12}, \ldots\ldots\ldots, f_{n2}, f_0$, whose coefficients are independent of z_{ij} $(i=1, 2, \ldots\ldots, n;\ j=1, 2)$ and which

contains all the f's expricitly, the function $f((z_i))$ is said to have an *algebraic addition theorem.*

Our object is to find the form of analytic functions which have this property. The main result is:

Theorem. In order that an analytic function $f((z_i))$, defined all over the Gauss space, may have an algebraic addition theorem, it is necessary and sufficient that $f((z_i))$ should be an algebraic fuuction of

$$(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\cdots\cdots(z_{r_1}+a_{r_1})^{p_{r_1}},\ \cdots\cdots$$

$$(z_{r_{k-1}+1}+a_{r_{k-1}+1})^{p_{r_{k-1}+1}}\cdots\cdots(z_{r_k}+a_{r_k})^{p_{r_k}},\ e^{\alpha z_{r_k+1}\cdots\cdots z_{r_{k+1}}},\cdots\cdots$$

$$e^{\sigma z_{r_{s-1}+1}\cdots\cdots z_{r_s}},\ \wp(z_{r_s+1}\cdots\cdots z_{r_{s+1}},\omega_1^{(1)},\omega_2^{(1)}),\ \cdots\cdots,$$

where p's are integers, positive or negative, the coefficients as well as a_1, a_2, ..., a_{r_k}, α, ..., σ, $\omega_1^{(i)}$, $\omega_2^{(i)}$, $(i=1, 2, \ldots\ldots)$ are independent of the variables, and a_{r_i+1}, a_{r_i+2}, ..., $a_{r_{i+1}}$ $(i=0, 1, \ldots\ldots, k-1;\ r_0=0)$ are or are not zero simultaneously.

1. *Lemma 1. Let a function $\varphi(z)$ be analytic in the whole Gauss plane. For any two arguments z_1, z_2, if there exists a relation*

$$G(\varphi(z_1),\ \varphi(z_2),\ z_1+z_2)=0 \tag{2}$$

where $G(u, v, w)$ is an irreducible polynominal of u and v and is analytic with respect to w in the whole plane, then $\varphi(z)$ is either

i) *an algebraic function of z; or*

ii) *an algebraic function of $e^{\alpha z}$; or*

iii) *an algebraic function of $\wp(z, \omega_1, \omega_2)$.*

In virtue of (2), the analytic function $\varphi(z)$ has only a finite number of branch points in any finite part of the plane. Accordingly z_1 and z_2 $(z_1+z_2=\text{constant})$ may describe certain closed circuits, by which $\varphi(z_2)$ returns to its initial element while $\varphi(z_1)$ changes into any other element. As (2) always holds for continuations and G is a polinominal of $\varphi(z_1)$, the number of all branches of $\varphi(z_1)$ is finite, i.e. *$\varphi(z)$ is a finitely many-valued function of z.* Let these branches be $\varphi_1(z)$, $\varphi_2(z)$, ..., $\varphi_n(z)$. Eliminating these values from

$$G(\varphi_i(z_1),\ \varphi(z_2),\ z_1+z_2)=0,\ (i=1, 2, \ldots\ldots, n)$$

and

$$S_1(\varphi_1(z_1),\ \varphi_2(z_1),\ \ldots\ldots,\ \varphi_n(z_1))\equiv S_1(\varphi(z_1))$$

where S_1 is any symmetric polinominal of φ_1, φ_2, ..., φ_n, we have

$$G_1(S_1(\varphi(z_1)),\ \ \varphi(z_2),\ \ z_1+z_2)=0,$$

where G_1 is of the same character as G. Again, eliminating $\varphi_1(z_2)$, $\varphi_2(z_2)$, ..., $\varphi_n(z_2)$ from

$$G_1(S_1(\varphi(z_1)),\quad \varphi_i(z_2),\quad z_1+z_2)=0$$

and

$$S_2(\varphi_1(z_2),\ \varphi_2(z_2),\ \ldots\ldots,\ \varphi_n(z_2))\equiv S_2(\varphi(z_2)),$$

where S_2 is any symmetric polinominal (identical to or different from S_1) of $\varphi_1(z_2)$, $\varphi_2(z_2)$, ..., $\varphi_n(z_2)$, we have

$$(3)\qquad G_2(S_1(\varphi(z_1)),\quad S_2(\varphi(z_2)),\quad z_1+z_2)=0,$$

where G_2 is of the same character as G and G_1. In G_2, $S_1(\varphi(z))$ and $S_2(\varphi(z))$ are evidently uniform functions of z. Putting $S_1\equiv S_2\equiv S$, we have

$$G_2(S(\varphi(z_1)),\quad S(\varphi(z_2)),\quad z_1+z_2)=0.$$

This is an extension of the assumption of Weierstras's theorem[1] and the same result follows by a similar proof. Put

$$(\varphi-\varphi_1)(\varphi-\varphi_2)\ldots\ldots(\varphi-\varphi_n)\equiv\varphi^n+S^{(1)}\varphi^{n-1}+S^{(2)}\varphi^{n-2}+\ldots\ldots+S^{(n)}.$$

Then $S^{(i)}$, $(i=1, 2, \ldots\ldots, n)$, is a symmetric polinominal of $\varphi_1, \varphi_2, \ldots\ldots, \varphi_n$. Put $S=S^{(i)}$ in (3), and we have the result that $S^{(i)}$ is a rational function of z, or of $e^{\alpha z}$, or of $\wp(z, \omega_1, \omega_2)$ and $\wp'(z, \omega_1, \omega_2)$. Again, putting $S_1=S^{(i)}$ and $S_2=S^{(j)}$, $(i\neq j)$, in (3), it follows easily that $S^{(i)}$ and $S^{(j)}$ are of the same character, i.e. they are rational functions of z, or of $e^{\alpha z}$, or of $\wp(z, \omega_1, \omega_2)$ and $\wp'(z, \omega_1, \omega_2)$ simultaneously. Hence $\varphi(z)$ is

i) an algebraic function of z; or

ii) an algebraic function of $e^{\alpha z}$; or

iii) an algebraic function of $\wp(z, \omega_1, \omega_2)$.

2. *Lemma 2. Let $D(z)$ be a finite domain of z and $E(z, z')$ be a field such that z is any point in D while z' is any finite point in the whole Gauss plane. If $\varphi(z, z')$ be a function which satisfies the conditions:*

i) *regular at any point in E,*

ii) *entire function at most of a finite order ρ in z' for any fixed value z in D,*

iii) *uniformly increasing function of z' for all z in D,*[2]

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[1] Osgood:—Lehrbuch der Funktionentheorie I. Zweite Auflage, p. 492.

[2] If, for any prescribed positive value ε, there corresponds a value R such that $|\varphi(z, z')|\leq e^{r^{\rho+\varepsilon}}$ for $|z'|\geq R$ and for all z in D, $\varphi(z, z')$ is said to be *uniformly increasing function* of z' for all z in D.

$\frac{\partial^s \varphi(z, z')}{\partial z^s}$ *is an entire function, at most, of the* p^{th} *order in* z' *for the same value of* z, *where* s *is any positive integer.*

Let

$$(4) \qquad \varphi(z, z') \equiv a_0(z) + a_0(z) + a_1(z)z' + \cdots\cdots + a_n(z)z'^n + \cdots\cdots$$

where $a_n(z)$, $(n=0, 1, \ldots\ldots)$ are regular functions of z in D, and $z=a$ be any point in D. Then since $a_n(z)$ is regular in D, it may be expressed as a power series of $(z-a)$ in the vicinity of $z=a$ so that we have

$$(5) \qquad a_n(z) \equiv \beta_{n0} + \beta_{n1}(z-a) + \cdots\cdots + \beta_{nm}(z-a)^m + \cdots\cdots$$

for $|z-a| \leqq R$, where β_{nm}, $(m=0, 1, 2, \ldots\ldots)$, are constants, and the domain represented by $|z-a| \leqq R$ is entirely in D. Let M be the maximum value of $|\varphi(z, z')|$ for $|z-a| \leqq R$ and $|z'| \leqq R'$. Then M is finite so far as R' is finite, and we have evidently

$$|a_n(z)| \leqq \frac{M}{R'^n} \quad \text{in} \quad |z-a| \leqq R, \quad (n=0, 1, 2, \ldots\ldots).$$

For any positive value ε, there corresponds a finite positive value N such that

$$|a_n(z)z'^n + a_{n+1}(z)z'^{n+1} + \cdots\cdots| \leqq \frac{Mr'^n}{R'^n}\left(1 + \frac{r'}{R'} + \cdots\cdots\right) \frac{Mr'^n}{R'^n\left(1 - \frac{r'}{R'}\right)} \leqq \varepsilon$$

for $n \geqq N$, in which $|z'| = r' < R'$ and $|z-a| \leqq R$.

Consequently, (4) is a uniformly convergent series of z in the domain represented by $|z-a| \leqq R$, for any arbitrarily assigned value of z' such as $|z'| < R'$. As R', is however, arbitrary, (4) is a uniformly convergent series of z in the domain such that $|z-a| \leqq R$, for any arbitrarily assigned value of z'. Moreover $a_n(z)$ is a regular function of z in D, and accordingly $\varphi(z, z')$ is termwise differentiable with respect to z.

$$(6) \qquad \frac{\partial^s \varphi(z, z')}{\partial z} = \frac{\partial_s a_0(z)}{\partial z^s} + \frac{\partial^s a_1(z)}{\partial z^s} z' + \cdots\cdots + \frac{\partial^s a_n(z)}{\partial z^s} z'^n + \cdots\cdots.$$

But we have from (5)

$$|\beta_{nm}| \leqq \frac{M}{R'^n} \cdot \frac{1}{R^m} = \frac{M}{R^m R'^n},$$

so that

$$\left|\frac{\partial^s a_n(z)}{\partial z^s}\right| \leqq s(s-1)\cdots\cdots 1\,|\beta_{ns}| + (s+1)s\cdots\cdots 2\,|\beta_{n,s+1}|z-a| + \cdots\cdots$$

$$\leqq \frac{M}{R^s R'^n}\cdot\frac{s!}{\left(1-\frac{r}{R}\right)^{s+1}}$$

$$= \frac{s!\,MR}{R'^n(R-r)^{s+1}},$$

where $|z-a| = r < R$. Hence, we have by (6)

$$\left|\frac{\partial^s \varphi(z, z')}{\partial z^s}\right| \leqq \sum_{n=0}^{\infty}\left|\frac{\partial^s a_n(z)}{\partial z^s}\right| |z'^n|$$

$$\leqq \frac{s!\,MR}{(R-r)^{s+1}}\sum_{n=0}^{\infty}\frac{r'^n}{R'^n}$$

$$= \frac{s!\,MRR'}{(R-r)^{s+1}(R'-r')}.$$

Put $|z'| = r' = R' - \sigma$ where σ is a certain positive constant. Then

$$\left|\frac{\partial^s \varphi(z, z')}{\partial z^s}\right| \leqq \frac{s!\,M(r'+\sigma)R}{(R-r)^{s+1}\sigma}.$$

Since $\varphi(z, z')$ is, by assumption, a uniformly increasing entire function of the p^{th} order in z' for all z in D, for any positive value ε, there is a corresponding value G such that $M < e^{R'^{p+\varepsilon}}$ [1] for all $R' \geqq G$ and for all z in D. Accordingly, for any fixed value of z in D, we have

$$\left|\frac{\partial^s \varphi(z, z')}{\partial z^s}\right| \leqq e^{R'^{p+\varepsilon}}\cdot\frac{s!\,R(r'+\sigma)}{(R-r)^{s+1}\sigma} = \frac{s!\,R(r'+\sigma)}{(R-r)^{s+1}\sigma}\cdot e^{r'^{p+\varepsilon}\left(1+\frac{\sigma}{r'}\right)^{p+\varepsilon}} \leqq e^{r'^{p+\varepsilon'}}$$

where ε' is so chosen as to satisfy

$$\frac{s!\,R}{(R-r)^{s+1}\sigma}(r'+\sigma) \leqq e^{r'^{p+\varepsilon}\left(r'^{\varepsilon'-\varepsilon}-\left(1+\frac{\sigma}{r'}\right)^{p+\varepsilon}\right)}.$$

---

[1] Let $M(R')$ be the maximum value of $|\varphi(z, z')|$ for $|z'| = R'$, z being an arbitrarily fixed value in D. Then, as $\varphi(z, z')$ is uniformly increasing, we have $M(R') < e^{R'^{p+\varepsilon}}$ for $R' \geqq G$ and for all z in D. Since $\varphi(z, z')$ is, however, a holomorphic function of z' for any arbitrarily assigned value of z, we have $M = M(R')$, so that $M < e^{R'^{p+\varepsilon}}$ for all $R' \geqq G$ and for all z in D.

By taking $R'=r'+\sigma$ sufficiently great, we may take ε and ε' as small as we please. Hence the proposition is true.

Lemma 3. Let

$$\Psi(z, z') \equiv \frac{\varphi_1(z, z')}{\varphi_2(z, z')}$$

where $\varphi_1(z, z')$ *and* $\varphi_2(z, z')$ *are functions which satisfy the conditions* i), ii), *and* iii) *in lemma 2, and are of the orders* p_1 *and* p_2 *in* z' *respectively. Then* $\frac{\partial^s \Psi(z, z')}{\partial z^s}$ (s: *any positive integer*), *is of the order at most equal to the greater of* p_1 *and* p_2.

Supposing the greater of p_1 and p_2 be p, $\Psi(z, z')$ is a meromorphic function, at most, of the p^{th} order in z' for any assigned value of z in D.

$$\frac{\partial \Psi}{\partial z} = \frac{\varphi_2 \frac{\partial \varphi_1}{\partial z} - \varphi_1 \frac{\partial \varphi_2}{\partial z}}{\varphi_2^{\,2}}$$

By lemma 2, $\frac{\partial \varphi_1}{\partial z}$ and $\frac{\partial \varphi_2}{\partial z}$ are entire functions, at most, of the p^{th} order in z', and accordingly $\frac{\partial \Psi}{\partial z}$ is a mermorphic function, at most, of the p^{th} order in z'. We may easily prove, by mathematical induction, that $\frac{\partial^s \Psi(z, z')}{\partial z^s}$ is a meromorphic function, at most, of the p^{th} order in z', for any arbitrarily assigned value of z in D, where s is any positive integer.

3. We proceed now to prove the theorem. First, we prove that the condition is sufficient.

Let $f((z_i))$ be an algebraic function of $(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\ldots(z_{r_1}+a_{r_1})^{p_{r_1}}$, ..., $(z_{r_{k-1}+1}+a_{r_{k-1}+1})^{p_{r_{k-1}+1}}\ldots\ldots\ldots(z_{r_k}+a_{r_k})^{p_{r_k}}$, $e^{\alpha z_{r_k+1}\ldots\ldots z_{r_{k+1}}}$, ..., $e^{\sigma z_{r_{s-1}+1}\ldots\ldots z_{r_s}}$, $\wp(z_{r_s+1}\ldots\ldots z_{r_{s+1}}, \omega_1^{(i)}, \omega_2^{(i)})$, ..., where p's are integers, positive or negative, the coefficients as well as a_1, a_2, ..., a_{r_k}, a, ..., σ, $\omega_1^{(i)}$, $\omega_2^{(i)}$, ($i=1, 2, \ldots\ldots$), are independent of the variables, and $a_{r_i+1}, a_{r_i+2}, \ldots\ldots, a_{r_{i+1}}$, ($i=0, 1, \ldots\ldots, k-1$; $r_0=0$), are or are not zero simultaneously.

We denote $z_1, z_2, \ldots\ldots, z_{r_1}$ by Γ_1, $z_{r_1+1}, z_{r_1+2}, \ldots\ldots, z_{r_2}$ by Γ_2 and so on. Then the number p of all Γ's thus obtained is evidently

less than or equal to n. Put

$$(7)\quad \begin{cases} (z_{11}+z_{12}+a_1)^{p_1}(z_{21}+z_{22}+a_2)^{p_2}\ \ldots\ldots\ldots(z_{r_1 1}+z_{r_2 2}+a_{r_1})^{p_{r_1}}\equiv\xi_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (z_{r_{k-1}+11}+z_{r_{k-1}+12}+a_{r_{k-1}+1})^{p_{r_{k-1}+1}}\ldots\ldots\ldots(z_{r_k 1}+z_{r_k 2}+a_{r_k})^{p_{r_k}}\equiv\xi_k \\ e^{\sigma(z_{r_k+11}+z_{r_k+12})}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots

Eliminating $\dfrac{\xi_1}{(z_{i1}+z_{i2}+a_i)^{p_i}}$, $(i=1, 2, \ldots\ldots, r_1)$, from (9) and the first equation in (7), we have

$$(10) \qquad \Phi_1(f_{11}, f_{12}, \ldots\ldots, f_{r_1 2}, \xi_1, \xi_2, \ldots\ldots, \xi_p)=0$$

where Φ_1 is an irreducible polynominal of the arguments and contains f_{ij}, $(i=1, 2, \ldots\ldots, r_1; j=1, 2)$, and ξ_1 explicitly. The same reasoning holds for $\Gamma_2, \ldots\ldots, \Gamma_k$, and we have

$$(11) \qquad \Phi_i=0, \quad (i=1, 2, \ldots\ldots, k),$$

where Φ_i is of the form (9) or (10).

As f_{ij}, $(i=r_k+1, \ldots\ldots, r_{k+1}; j=1, 2)$, is an algebraic function of $\xi_1, \xi_2, \ldots\ldots, \xi_k, \xi_{k+2}, \ldots\ldots, \xi_p$ and $e^{\alpha(z_{r_k+11}+z_{r_k+12})\ldots\ldots(z_{i-11}+z_{i-12})z_{ij}(z_{i+11}+z_{i+12})\ldots\ldots(z_{r_{k+1}1}+z_{r_{k+1}2})}$, we may eliminate $e^{Az_{i1}}$ and $e^{Az_{i2}}$ from these two equations for $j=1, 2$ and $e^{Az_{i1}} \cdot e^{Az_{i2}}=e^{A(z_{i1}+z_{i2})}=\xi_{k+1}$, where

$$A \equiv \alpha(z_{r_k+11}+z_{r_k+12})\ldots\ldots(z_{i-11}+z_{i-12})(z_{i+11}+z_{i+12})\ldots\ldots (z_{r_{k+1}1}+z_{r_{k+1}2})$$

and we have

$$(12) \quad \Phi_{k+1}(f_{i1}, f_{i2}, \xi_1, \xi_2, \ldots\ldots, \xi_k, \xi_{k+1}, \xi_{k+2}, \ldots\ldots, \xi_p)=0, \quad (i=r_k+1, \ldots\ldots, r_{k+1}),$$

where Φ_{k+1} is an irreducible polynominal of the arguments and contains $f_{i1}, f_{i2}, \xi_{k+1}$ explicitly. The same reasoning holds for $\Gamma_{k+2}, \ldots\ldots, \Gamma_s$ and we have

$$(13) \quad \Phi_i(f_{j1}, f_{j2}, \xi_1, \xi_2, \ldots\ldots, \xi_p)=0, \quad (i=k+1, \ldots\ldots, s; j=\text{any one of } r_{i-1}+1, \ldots\ldots, r_i),$$

where Φ_i is, as Φ_{k+1}, an irreducible polynomial of the arguments and contains f_{j1}, f_{j2}, ξ_i explicitly.

Let

$$\Psi(\wp(u), \quad \wp(v), \quad \wp(u+v))=0$$

be an algebraic addition-theorem of $\wp$-function. Then we have

$$(14) \quad \Psi(\wp(Bz_{i1}), \wp(Bz_{i2}), \xi_{s+1})=0$$

where i is any one of $r_s+1, \ldots\ldots, r_{s+1}$ and

$$B\equiv(z_{r_s+11}+z_{r_s+12})\ldots\ldots(z_{i-11}+z_{i-12})\,(z_{i+11}+z_{i+12})\ldots\ldots(z_{r_{s+1}{}^1}+z_{r_{s+1}{}^2}),$$
$$\xi_{s+1}\equiv\wp((z_{r_s+11}+z_{r_s+12})\ldots\ldots\ldots\ldots(z_{r_{s+1}{}^1}+z_{r_{s+1}{}^2})).$$

As f_{ij}, $(i=r_s+1, \ldots\ldots, r_{s+1};\ j=1, 2)$, is an algebraic function of $\xi_1, \xi_2, \ldots\ldots, \xi_s, \xi_{s+2}, \ldots\ldots, \xi_p$ and $\wp((z_{r_s+11}+z_{r_s+12})\ldots\ldots\,(z_{i-11}+z_{i-12})\times z_{ij}(z_{i+11}+z_{i+12})\ldots\ldots(z_{r_{s+1}{}^1}+z_{r_{s+1}{}^2}))$, we may eliminate $\wp(Bz_{i1})$ and $\wp(Bz_{i2})$ from these two equations for $j=1, 2$ and (14), and we have

$$(15)\quad \Phi_{s+1}(f_{i1}, f_{i2}, \xi_1, \ldots\ldots, \xi_s, \xi_{s+1}, \xi_{s+2}, \ldots\ldots, \xi_p)=0,$$
$$(i=r_s+1, \ldots\ldots, r_{s+1}),$$

where Φ_{s+1} is an irreducible polynominal of the arguments and contains $f_{i1}, f_{i2}, \xi_{s+1}$ explicitly.

The same reasoning holds for $\Gamma_{s+2}, \ldots\ldots, \Gamma_p$ and we have

$$(16)\quad \Phi_i(f_{j1}, f_{j2}, \xi_1, \xi_2, \ldots\ldots, \xi_p)=0,$$
$$(i=s+1, \ldots\ldots, p;\ j=\text{any one of } r_{i-1}+1, \ldots\ldots, r_i),$$

where Φ_i is, as Φ_{s+1}, an irreducible polynominal of the arguments and contains f_{j1}, f_{j2}, ξ_i explicitly.

As $f_0\equiv f(z_{11}+z_{12}, \ldots\ldots, z_{n1}+z_{n2})$ is an algebraic function of $\xi_1, \xi_2, \ldots\ldots, \xi_p$ we have

$$(17)\quad \Phi_{p+1}(f_0, \xi_1, \xi_2, \ldots\ldots, \xi_p)=0$$

where Φ_{p+1} is an irreducible polynominal of the arguments $f_0, \xi_1, \xi_2, \ldots\ldots, \xi_p$ and contains all of them explicitly.

5. Eliminating $\xi_1, \xi_2, \ldots\ldots, \xi_p$ from $p+1$ equations in (11), (13), (16) and (17), we have

$$(18)\quad \Psi_1=0$$

where Ψ_1 is an irreducible polynominal of the arguments. If one of f_{i1} and f_{i2} be not contained in Ψ_1, then the other will not be also. This may be shown as in the preceeding article.

The case where Ψ_1 contains none of f_{i1} and f_{i2}, $(i=$any one of $r_{t-1}+1, \ldots\ldots, r_t)$, will occur when and only when ξ_t may be also eliminated by the process of elimination of $\xi_1, \xi_2, \ldots\ldots, \xi_{t-1}, \xi_{t+1}, \ldots\ldots, \xi_p$ from $\Phi_1=0, \ldots\ldots, \Phi_{t-1}=0, \Phi_{t+1}=0, \ldots\ldots, \Phi_{p+1}=0$ (without using $\Phi_t=0$). In this case we first eliminate ξ_t from $\Phi_i=0$ and $\Phi_{p+1}=0$ and then

eliminate the other ξ's. Thus we have, at least, one irreducible factor of the resultant, which contains f_{i1} and f_{i2}, (i=any one of $r_{t-1}+1$, ..., r_t), explicitly. Let one of them be Ψ_2, and we have,

(19) $\Psi_2=0$.

If there is at least one Γ such that all f_{ij} which correspond to it are contained in neither Ψ_1 nor Ψ_2, then we have, by repeating the same reasoning,

(20) $\Psi_i=0$, ($i=1, 2, \ldots\ldots, q$; $q\leqq p$),

where Ψ_i, ($i=1, 2, \ldots\ldots, q$), is irreducible. Hence, there corresponds, to each Γ_t, at least one Ψ_i which contains some of the f's corresponding to that Γ_t.

6. *For suitably chosen constants* $\lambda_2, \lambda_3, \ldots\ldots, \lambda_q$, *all different from zero,*

$$\Psi_1+\lambda_2\Psi_2+\ldots\ldots+\lambda_q\Psi_q$$

is an irreducible polynomial of all f*'s which are contained at least in one of* $\Psi_1, \Psi_2, \ldots\ldots, \Psi_q$ *and contains all of them explicitly.*

For, let one of the f's, say f_{11}, be contained in either or both of Ψ_1 and Ψ_2 and suppose that Ψ_1 and Ψ_2 are polpnomials of the m_1^{th} and m_2^{th} degrees in f_{11} respectively. Then we may assume, without loss of generality, that $m_1\geqq m_2$. We now assume that there are infinitely many values of μ in the vicinity of $\mu=0$ for which $\Psi_1+\mu\Psi_2$ is reducible, i.e.

$$\Psi_1+\mu\Psi_2\equiv(M_m f_{11}^m+\ldots\ldots+M_0)(N_{m'}f_{11}^{m'}+\ldots\ldots+N_0)$$

where m and m' are zero or positive integers such that $m+m'=m_1$, and $M_0, \ldots\ldots, M_m, N_0, \ldots\ldots, N_{m'}$ are polynomials of f's except f_{11}, contained in Ψ_1, Ψ_2. Let those values of μ be $\mu_1, \mu_2, \ldots\ldots, \mu_i, \ldots\ldots$. Then $\lim_{i=\infty}\mu_i=0$. To each μ_i, there correspond $m(\geqq 0)$ and $m'(\geqq 0)$ such that $m+m'=m_1$, and the number of combinations of (m, m') is finite while the number of the values of μ_i is infinite. Accordingly there are infinitely many values of μ_i which correspond to one and the same combination (m, m'). Considering the limit of these values of μ_i for $i=\infty$, we have

$$\Psi_1\equiv(M_m f_{11}^m+\ldots\ldots+M_0)(N_{m'}f_{11}^{m'}+\ldots\ldots+N_0).$$

As Ψ_1 is, however, irreducible, one of m and m', say m, is equal to m_1, and N_0 is an absolute constant. Hence there is at most only a finite

number of μ's in the vicinity of $\mu=0$, for which $\Psi_1+\mu\Psi_2$ is reducible, i.e. $\Psi_1+\mu\Psi_2$ becomes irreducible for infinitely many values of μ. There exists at most only a finite number of values of μ, for which $\Psi_1+\mu\Psi_2$ does not contain explicitly at least one of the f's contained in Ψ_1, Ψ_2. Consequently, there are infinitely many values of μ, such that $\Psi_1+\mu\Psi_2$ is irreducible and contains all the f's explicitly, which are contained in Ψ_1, Ψ_2. By repeating the same reasoning, we have the proposition.

7. Put

$$\Psi_1+\lambda_2\Psi_2+\cdots\cdots\cdots+\lambda_q\Psi_q\equiv X_1.$$

Then X_1 is an irreducible polynomial of all the f's contained in Ψ_1, Ψ_2, ..., Ψ_q and contains all of them explicitly. Accordingly X_1 contains some of the f's explicitly, which correspond to any one of Γ_1, Γ_2, ..., Γ_p.

$$(21)\quad X_1=0.$$

Suppose that f_{i1} and f_{i2} be contained in X_1 and that z_j be any other variable corresponding to the same Γ_t as z_i $(i\gtrless j)$. X_1 contains f_{j1} and f_{j2} when and only when Γ_t is some one of Γ_1, Γ_2, ..., Γ_k and Φ_t has the form as (10). If X_1 does not contain f_{j1} and f_{j2}, we substitute f_{j1} and f_{j2} instead of f_{i1} and f_{i2} respectively. By all possible such substitutions, we obtain X_1, X_2, ..., X_l, where $l\leqq r_1(r_2-r_1)$ ....., $\times(r_{s+1}-r_s)$..... All the f's are now contained in X_1, X_2, ..., X_l.

$$(22)\quad X_i=0,\quad (i=1,\ 2,\ \cdots\cdots,\ l).$$

By the same reasoning as in the preceding article, there are $(l-1)$ constants μ_2, μ_3, ..., μ_l, all different from zero, such that

$$X_1+\mu_2X_2+\cdots\cdots\cdots+\mu_tX_t$$

is irreducible and contains all the f's explicitly.

Let

$$X_1+\mu_2X_2+\cdots\cdots\cdots+\mu_tX_t\equiv F(f_{11},\ f_{12},\ f_{21},\ \cdots\cdots,\ f_{n2},\ f_0)=0.$$

Then *F is an irreducible polynominal of f_{11}, f_{12}, f_{21}, ..., f_{n2}, and f_0 whose coefficients are independent of z_{11}, z_{12}, z_{21}, ..., z_{n2} and contains all the f's explicitly. Hence $f((z_i))$ has an algebraic addition-theorem.*

8. We now proceed to prove that the condition is necessary. Since $f((z_i))$ is, by assumption, analytic function of z_1, z_2, ..., z_n, defined

all over the Gauss space, we may choose a set of fixed points $z_{ij}=a_{ij}$, $(i=1, 2, \ldots\ldots, n;\ j=1, 2)$ such that

$$f_0\equiv f(z_{11}+z_{12}, \ldots\ldots\ldots\ldots, z_{n1}+z_{n2}) \qquad \text{and}$$

$$f_{ij}\equiv f(z_{11}+z_{12}, \ldots\ldots, z_{i-11}+z_{i-12}, z_{ij}, z_{i+11}+z_{i+12}, \ldots\ldots, z_{n1}+z_{n2}).$$
$$(i=1, 2, \ldots\ldots, n;\ j=1, 2),$$

are analytic in the vicinity of $z_{ij}=a_{ij}$ and satisfy the equation

$$(1)\quad F(f_{11}, f_{12}, f_{21}, \ldots\ldots\ldots, f_{n2}, f_0)=0;$$

that is to say, since $f((z_i))$ is generally a many-valued function, (1) will be satisfied by certain branches of f_0 and of f_{ij}, where the branches, being suitably chosen, are regular functions of z_{ij}'s in the vicinity of $z_{ij}=a_{ij}$. As $f((z_i))$ is an analytic function defined all over the Gauss space, we may easily prove, from (1), that $f((z_i))$ has only a finite number of branch points in any finite part of the plane, with respect to any one of the independent variables, the other ones being considered as arbitrary constants. Moreover, for any assigned values of $z_{21}, z_{22}, \ldots\ldots\ldots, z_{n2}$, we may select closed circuits in the planes of z_{11} and z_{12} respectively, along which the functions are continuated, such that all f's except f_{11} return to their initial elements while f_{11} changes into any other element. Proceeding in the same way as in the proof of lemma 1, we may prove that the number of all branches of f_{11}, considered as a function of z_{11} only, is finite; that is, $f((z_i))$ is a finitely many-valued function of z_1. Similarly for $z_2, z_3, \ldots\ldots\ldots, z_n$. Consequently, $f((z_i))$ is a finitely many-valued function of $z_1, z_2, \ldots\ldots\ldots, z_n$.

For arbitrarily assigned values of z_{ij}, $(i=2, 3, \ldots\ldots\ldots, n;\ j=1, 2)$, we may write $f_{11}\equiv\varphi(z_{11})$, $f_{12}\equiv\varphi(z_{12})$ and $F\equiv G(\varphi(z_{11}), \varphi(z_{12}), z_{11}+z_{12})=0$, where $\varphi(z)$ is an analytic function of z in the whole Gauss plane, and $G(u, v, w)$ is an irreducible polynominal of u and v and is analytic with respect to w in the whole Gauss plane. Then, by lemma 1 in §1, $f((z_i))$ is an algebraic function of z_1, or of $e^{\alpha z_1}$, or, of $\wp(z_1, \omega_1, \omega_2)$ where the coefficients as well as a, ω_1, ω_2 are generally functions of $z_2, z_3, \ldots\ldots\ldots, z_n$. We may, however, assume that $f((z_i))$ is an algebraic function of $\wp(z_1, \omega_1, \omega_2)$. For, if the invariants g_2, g_3 satisfy the condition that $\triangle\equiv g_2^3-27g_3^2=0$, then $f((z_i))$ is an algebraic function of $e^{\alpha z_1}$; if $g_2=g_3=0$, then $f((z_i))$ is an algebraic function of z_1. The same reasoning holds for $z_2, z_3, \ldots\ldots\ldots, z_n$. Accordingly $f((z_i))$ is an algebraic function of $\wp(z_1, \omega_1, \omega_2)$ and is, at the same time, an algebraic function of $\wp(z_2, \Omega_1, \Omega_2)$.

9. We assume that at least one of ω_1 and ω_2 is dependent of z_2 and that at least one of Ω_1 and Ω_2 is dependent of z_1. For simplicity, we use the notations z, z' instead of z_1, z_2 respectively and consider z_3, ..., z_n as parameters. In the vicinity of the origin suitably chosen, each branch of $f(z, z')$ is expansible as follows.

$$(23)\quad f(z, z') = a_0(z') + a_1(z')z + \ldots\ldots + a_r(z')z^r + \ldots\ldots$$

$$(24)\qquad = A_0(z) + A_1(z)z' + \ldots\ldots + A_r(z)z'^r + \ldots\ldots$$

where $a_r(z')$ and $A_r(z)$, $(r=0, 1, \ldots\ldots)$, are regular in the vicinity of the origin. From what has been proved (§8), it follows that $f(z, z')$ may also satisfy the irreducible equations

$$(25)\quad \left\{b_{m m_m}(z')\wp^{m_m}(z, \omega_1(z'), \omega_2(z')) + \ldots\ldots + b_{m0}(z')\right\}f^m(z, z') + \ldots\ldots$$
$$+ \left\{b_{0m_0}(z')\wp^{m_0}(z, \omega_1(z'), \omega_2(z')) + \ldots\ldots + b_{00}(z')\right\} = 0,$$
$$(b_{mm_m}(z') \equiv),$$

$$(26)\quad \left\{B_{nn_n}(z)\wp^{n_n}(z', \Omega_1(z), \Omega_2(z)) + \ldots\ldots B_{n0}(z)\right\}f^n(z, z') + \ldots\ldots$$
$$+ \left\{B_{0n_0}(z)\wp^{n_0}(z', \Omega_1(z), \Omega_2(z)) + \ldots\ldots + B_{00}(z)\right\} = 0, (B_{nn_n}(z) \equiv 1),$$

where $b(z')$'s and $B(z)$'s are functions of z' and z respectively. Here we remark that the relation between z and z' is quite reciprocal. If a proposition is made concerning z, z', the same may be directly derived only by interchanging the corresponding functions. Hence, to simplify this relation, we use for the corresponding elements the letters a, A; b, B; m, n.

Substitute the power series in (23) and

$$\frac{1}{z^2} + \frac{g_2(z')}{20}z^2 + \frac{g_3(z')}{28}z^4 + \ldots\ldots$$

instead of $f(z, z')$ and $\wp(z, \omega_1(z'), \omega_2(z'))$ in (25) respectively, and equate the coefficient of each power of z to zero. Then an infinite number of linear homogeneous equations among $b(z')$'s, whose coefficients are polynomials of $g_2(z')$, $g_3(z')$ and $a_r(z')$, $(r=0, 1, 2, \ldots\ldots)$, are obtained. Eliminating $b(z')$'s from these equations, we have algebraic relations among $g_2(z')$, $g_3(z')$ and $a(z)$'s. Again eliminating $g_2(z')$ or $g_3(z')$, we may express $g_2(z')$ and $g_3(z')$ as algebraic functions of a finite number of $a(z)$'s. This process may be effected without fail. The only ex-

ceptional case to be considered is that in which $g_2(z')$ and $g_3(z')$ are eliminated always at the same time. In this case, one of $g_2(z')$ and $g_3(z')$ and accordingly one of $\omega_1(z')$ and $\omega_2(z')$ may be assigned arbitrarily and the other may be determined so as to satisfy (25). Let $\omega_1^{(1)}(z')$ and $\omega_1^{(2)}(z')$ be two values of $\omega_1(z')$, and $\omega_2^{(1)}(z')$ and $\omega_2^{(2)}(z')$ be their corresponding values of $\omega_2(z')$ respectively. Now we have two equations of the form (25), which will be satisfied by the same value of $f(z, z')$. Hence there exists, between $\wp(z, \omega_1^{(1)}(z'), \omega_2^{(1)}(z'))$ and $\wp(z, \omega_1^{(2)}(z'), \omega_2^{(2)}(z'))$, an algebraic relation and accordingly $\omega_1^{(1)}(z')$, $\omega_2^{(1)}(z')$, $\omega_1^{(2)}(z')$, $\omega_2^{(2)}(z')$ are connected by

$$\begin{cases} p\omega_1^{(2)} = q\omega_1^{(1)} + r\omega_2^{(1)} \\ p'\omega_2^{(2)} = q'\omega_1^{(1)} + r'\omega_2^{(1)} \end{cases}$$

where p, q, r, p', q', r', are integers and $p \gtrless 0$, $p' \gtrless 0$. As $\omega_1^{(2)}(z')$ is arbitrary, let

$$\omega_1^{(2)}(z') \equiv k\omega_1^{(1)}(z')$$

where k is real but irrational. We have

$$(pk - q)\omega_1^{(1)}(z') = r\omega_2^{(1)}(z').$$

Consequently $f(z, z')$ would have an infinitesimal period, which is certainly impossible unless it is a constant. The invariants $g_2(z')$ and $g_3(z')$ are, therefore, algebraic functions of a finite number of $a_r(z')$, $(r = 0, 1, 2, \ldots\ldots)$. Since all $a_r(z')$, $(r = 0, 1, \ldots\ldots)$, behave regularly in the vicinity of the origin, $g_2(z')$ and $g_3(z')$ behave algebraically in the same region. By means of linear homogeneous equations of $b(z')$'s, $(b_{m m_m}(z') \equiv 1)$, whose coefficients are polynominals of $g_2(z')$, $g_3(z')$ and $a(z')$'s, $b(z')$'s are also algebraic in the same region. Thus, *if $f(z, z')$ be regular in the vicinity of the origin, then all functions $g_2(z')$, $g_3(z')$ and $b(z)$'s behave algebraically in the same region.*

A similar reasoning holds for $G_2(z)$, $G_3(z)$ and $B(z)$'s.

10. As $f(z, z')$ and $B(z)$'s are algebraic in the vicinity of the origin, the same is true, by (26), of $\wp(z', G_2(z), G_3(z))$. Hence, all branches of $\wp(z', G_2(z), G_3(z)) - \frac{1}{z'^2}$ are regular functions of z and z' in the vicinity of the origin which is suitably chosen. Introduce the functions

$$\zeta(z')=-\int p(z)dz'+C_1, \qquad \left(\lim_{z'=0}\left(\zeta(z')-\frac{1}{z'}\right)=0\right),$$

$$\sigma(z')=e^{\int \zeta(z')dz'+C_2}, \qquad \left(\lim_{z'=0}\frac{\sigma(z')}{z'}=1\right).$$

Then all branches of $\sigma(z', G_2(z), G_3(z))$ are regular in the same field of variations. Moreover

$$\sigma(z', G_2(z), G_3(z))=z'\prod\left(1-\frac{z'}{m\Omega_1(z)+n\Omega_2(z)}\right) \times e^{\frac{z'}{m\Omega_1(z)+n\Omega_2(z)}+\frac{z'^2}{2(m\Omega_1(z)+n\Omega_2(z))^2}}$$

is a uniformly increasing entire function of the 2^{nd} order in z' for all z in the vicinity of the origin. Similarly, the same is true of

$$\Phi(z, z')\equiv\sigma^{2n_i}(z', G_2(z), G_3(z))\{B_i n_i(z)\wp^{n_i}(z', G_2(z), G_3(z))+\ldots\ldots\ldots\ldots B_{i0}(z)\},$$

where $(i=0, 1, \ldots\ldots\ldots\ldots, n)$,

$$(27)\quad B_i n_i(z)\wp^{n_i}(z', G_2(z), G_3(z))+\ldots\ldots\ldots\ldots B_{i0}(z)$$

is the coefficient of $f^{(i)}(z, z')$ in (26). Hence, the numerator and the denominator af each branch of $\Phi(z, z')/\sigma^{2n_i}(z, G_2(z), G_3(z))$ are regular functions of z and z' in the assigned field and are uniformly increasing entire functions of the 2^{nd} order in z' for all z in the vicinity of the origin. Hence by lemma 3, §2, the same is true of the derivatives of $\Phi(z, z')/\sigma^{2n_i}(z, G_2(z), G_3(z))$ with respect to z. Substitute the power series in (23) instead $f(z, z')$ in (26). Then, differentiating both members of the equation r-times with respect to z and assigning zero for z, we have $a_r(z')$, $(r=0, 1, 2, \ldots\ldots\ldots)$, as a transcendental algebraic function, at most, of the 2^{nd} order. Since $g_2(z')$, $g_3(z')$ are algebraic functions of a finite number of $a_r(z')$, $r=0, 1, 2, \ldots\ldots\ldots\ldots)$ (§9), *they are also transcendental algebraic functions, at most, of the* 2^{nd} *order. Similarly for* $b(z')$'s $(b_{mm_m}(z')\equiv 1)$.

The same reasoning holds for $G_2(z)$, $G_3(z)$ and $B(z)$'s $(B_{nn_n}(z)\equiv 1)$. We may further conclude that, by

$$\frac{g_3^2(z')}{g_2^3(z')-27g_3^2(z')}\equiv J(z')\equiv\frac{4}{27}\cdot\frac{\{1-\varkappa^2(z')+\varkappa^4(z')\}^3}{\varkappa^4(z')\cdot\{1-\varkappa^2(z')\}^2},$$

[1] If $g_2(z')\equiv g_3(z')\equiv 0$, $\varkappa^2(z')$ will be indeterminate. But in this case, we have $\omega_1(z')\equiv\omega_2(z')\equiv\infty$, i e. $\omega_1(z')$ and $\omega_2(z')$ are independent of z', which is against the assumption (§9).

$J(z')$ and $\varkappa^2(z')\equiv\zeta(z')$ are transcendental algebraic functions, at most, of the 2^{nd} order.

11. Let u_1, u_2 be a canonic fundamental system of the solutions of

$$(28)\quad \zeta(\zeta-1)\frac{d^2u}{d\zeta^2}+(2\zeta-1)\frac{du}{d\zeta}+\frac{1}{4}u=0.$$

Then they may be expressed as follows. In the vicinity of $\zeta=0$,

$$u_1^{(0)}=F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta\right)$$

$$u_2^{(0)}=F_1\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta\right)+F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta\right)\log\zeta;$$

in the vicinity of $\zeta=1$,

$$u_1^{(1)}=F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ 1-\zeta\right)$$

$$u_2^{(1)}=F_1\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ 1-\zeta\right)+F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ 1-\zeta\right)\log(1-\zeta);$$

in the vicinity of $\zeta=\infty$,

$$u_1^{(\infty)}=\left(\frac{1}{\zeta}\right)^{\frac{1}{2}}F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \frac{1}{\zeta}\right)$$

$$u_2^{(\infty)}=\left(\frac{1}{\zeta}\right)^{\frac{1}{2}}\left\{F_1\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \frac{1}{\zeta}\right)+F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \frac{1}{\zeta}\right)\log\left(\frac{1}{\zeta}\right)\right\},$$

where $F\left(\frac{1}{2},\frac{1}{2},1,x\right)$ and $F\left(\frac{1}{2},\frac{1}{2},1,x\right)$ are holomorphic functions of x in the vicinity of $x=0$, and $F\left(\frac{1}{2},\frac{1}{2},1,0\right)=1$. In the vicinity of any other point $\zeta=\alpha$, u_1 and u_2 are integral power series of $(\zeta-\alpha)$. Since u_1, u_2 are a fundamental system of the solutions, we may assume that $u_1(\alpha)\neq 0$, $u_2(\alpha)=0$, while $\frac{du_2(\alpha)}{d\alpha}\neq 0$. Then all solutions of (28) are expressible by linear combinations of u_1 and u_2.

Now, ζ is a modular function of τ which is a ratio of two linearly independent solutions of (28), so that there is a doubly periodic function $\wp(z, \bar{\omega}_1(z'), \bar{\omega}_2(z'))$ whose periods are solution of (28), the ratio being equal to τ. Hence, we may put

$$(29)\quad \begin{cases} \bar{\omega}_1(z') = Au_1\{\zeta(z')\} + Bu_2\{\zeta(z')\}, \\ \bar{\omega}_2(z') = Cu_1\{\zeta(z')\} + Du_2\{\zeta(z')\}, \end{cases}$$

where A, B, C, D are constants such that the imaginary part of $\frac{Cu_1+Du_2}{Au_1+Bu_2}$ is always positive and $AD-BC\neq 0$. We have, accordingly,

$$\tau(z') \equiv \frac{\bar{\omega}_2(z')}{\bar{\omega}_1(z')} \equiv \frac{Cu_1\{\zeta(z')\} + Du_2\{\zeta(z')\}}{Au_1\{\zeta(z')\} + Bu_2\{\zeta(z')\}}.$$

Let

$$\bar{g}_2(z') \equiv 60 \sum \frac{1}{\{m\bar{\omega}_1(z') + n\bar{\omega}_2(z')\}^4},$$

$$\bar{g}_3(z') \equiv 140 \sum \frac{1}{\{m\bar{\omega}_1(z') + n\bar{\omega}_2(z')\}^6},$$

where the summations exclude the simultaneous zero values of m and n in the expression of $m\bar{\omega}_1(z') + n\bar{\omega}_2(z')$. Then, as

$$-\frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{4}{27} \cdot \frac{(1-\zeta-\zeta^2)^3}{\zeta^2(1-\zeta)^2} = \frac{\bar{g}_2^3}{\bar{g}_2^3 - 27\bar{g}_2^3},$$

we have

$$g_2(z')\gamma^4(z') = \bar{g}_2(z')$$

and

$$g_3(z')\gamma^6(z') = \bar{g}_3(z'),$$

where $\gamma(z')$ is a certain function of z'.

Consequently, we may assume, without contradiction, that

$$\bar{\omega}_1(z')\gamma(z') = \omega_1(z'),$$

$$\bar{\omega}_2(z')\gamma(z') = \omega_2(z').$$

12. By (29), we see that if a circuit in the ζ-plane does not contain any one of the points 0, 1 and ∞, ω_1 and ω_2 return to their initial values after a complete rotation along the circuit. If the circuit contains, at least, one of the points 0, 1 and ∞, then at least one of ω_1 and ω_2 will change its value. As ζ is, however, a modular function of τ for group $\Gamma_6: \begin{pmatrix} p, & q \\ r, & s \end{pmatrix} \equiv \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} \bmod 2$, $\bar{\omega}_1$ and $\bar{\omega}_2$ will change into $\pm(r\bar{\omega}_2+s\bar{\omega}_1)$ and $\pm(p\bar{\omega}_2+q\bar{\omega}_1)$ respectively,[1] where p, q, r, s are integers such that $p\equiv 1 (\bmod 2)$, $q\equiv 0 (\bmod 2)$, $r\equiv 0 (\bmod 2)$, $s\equiv 1 (\bmod 2)$ and $ps-qr=1$. As $ps-qr=1$, $\bar{g}_2$ and $\bar{g}_3$ return to their initial values after a complete rotation of ζ along the circuit. Moreover $\zeta(z')$ is a transcendental algebraic function of z'. Hence, *$\bar{g}_2(z')$ and $\bar{g}_3(z')$, as functions of z', are finitely many-valued functions of z' except in the vicinity of the singular points.*

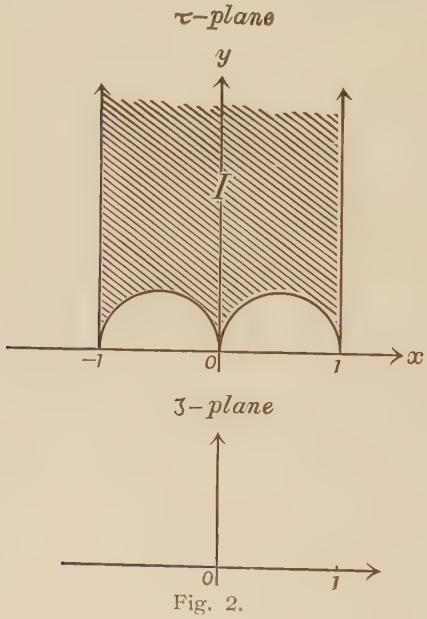

Fig. 2.

As $g_2(z')$ and $g_3(z')$ are, however, transcendental algebraic functions of z' (§10), they are finitely many-valued functions. Accordingly the same is true of $\gamma(z')$ except in the vicinity of the singular points of $\bar{g}_2(z')$ and $\bar{g}_3(z')$.

Since there exists a unique correspondency between the subdomain I of the τ-plane and the total ζ-plane, and since there correspond $\tau=-1$, 0 and ∞ to $\zeta=\infty$, 1 and 0 respectively (Fig. 2), $\tau(z')=\dfrac{\bar{\omega}_2(z')}{\bar{\omega}_1(z')}$ has a finite value from -1 and 0, at any finite point z' where $\zeta(z')\neq 0, 1, \infty$. Also

[1] If may be considered generally that $\bar{\omega}_1$ and $\bar{\omega}_2$ will change into $\bar{\omega}_1^{(1)}=\lambda r\bar{\omega}_2+\lambda s\bar{\omega}_1$ and and $\bar{\omega}_2^{(1)}=\lambda p\bar{\omega}_2+\lambda q\bar{\omega}_1$ respectively, where $\begin{pmatrix} p, & q \\ r, & s \end{pmatrix} \equiv \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$ mod 2, and $ps-qr=1$. But by the theory of automorphic functions, there correspond $\zeta=0$, $\zeta=1$ and $\zeta=\infty$ to $\tau=\infty$, $\tau=0$ and $\tau=-1$ respectively. Hence in the vicinity of $\zeta=1$, $\bar{\omega}_1=AF+B(F_1+F\log(1-\zeta))$, $\bar{\omega}_2=CF$, (§11), the constants A, B, C being suitably chosen. Accordingly $\lambda p=1$, $\lambda q=0$, $\lambda s=1$, so that $q=0$ and $\lambda p\cdot\lambda s=\lambda^2 ps=1$. We have, however, $ps-qr=ps=1$, and therefore $\lambda^2=1$ or $\lambda=\pm 1$. Consequently, we have $\bar{\omega}_1^{(1)}=\pm(r\bar{\omega}_2+s\bar{\omega}_1)$ and $\bar{\omega}_2^{(1)}=\pm(p\bar{\omega}_2+q\bar{\omega}_1)$.

Similarly the proposition is true in the vicinity of $\zeta=0$ and $\zeta=\infty$.

$\bar{\omega}_1(z')$, $\bar{\omega}_2(z')$ can not be zero simultaneously; for otherwise by (29). $\bar{\omega}_1 = Bu_2$ and $\bar{\omega}_2 = Du_2$ in the vicinity of the common zero point and accordingly $\tau = \dfrac{\bar{\omega}_2}{\bar{\omega}_1} = \dfrac{B}{D}$ would be a constant. Thus none of $\bar{\omega}_1$ and $\bar{\omega}_2$ becomes 0 or ∞ at any z' where $\zeta(z') \neq 0, 1, \infty$. Moreover, $\bar{\omega}_1(z')$ and $\bar{\omega}_2(z')$ are, by (29), algebraic in the vicinity of any finite point where $\zeta(z') \neq 0, 1, \infty$, and so also for $\bar{g}_2(z')$ and $\bar{g}_3(z')$. As $g_2(z')$ and $g_3(z)$ are, however, transcendental algebraic functions, $\gamma(z')$ is also algebraic in the vicinity of any finite point where $\zeta(z') \neq 0, 1, \infty$.

13. We assume that there is, at least, one finite point where $\zeta(z') = 0$. Let one of them be $z' = a$. As $\zeta(z')$ is a transcendental algebraic function, we may assign a neighborhood of $z' = a$, such that no other singular point of $\zeta(z')$ and no other point which satisfies $\zeta(z') = 0, 1$, or ∞ is found in it. In this domain, $\bar{g}_2(z')$ is algebraic except at $z' = a$. We have therefore

$$(30)\quad \bar{g}_2(z') = a_0 + a_1(z'-a)^{\frac{1}{\lambda}} + \cdots\cdots + a_i(z'-a)^{\frac{i}{\lambda}} + \cdots\cdots$$
$$+ a_{-1}(z'-a)^{-\frac{1}{\lambda}} + \cdots\cdots + a_{-i}(z'-a)^{-\frac{i}{\lambda}} + \cdots\cdots,$$

where λ is an integer. But in the vicinity of $\zeta(a) = 0$, $\bar{\omega}_1(z')$ and $\bar{\omega}_2(z')$ take the forms (§11)

$$\bar{\omega}_1(z') = AF\{\zeta(z')\},$$
$$\bar{\omega}_2(z') = DF_1\{\zeta('z)\} + F\{\zeta(z')\} \cdot \{C + D\log\zeta(z')\}.$$

Accordingly

$$\bar{g}_2(z') = 60 \sum \frac{1}{(m\bar{\omega}_1(z') + n\bar{\omega}_2(z'))^4}$$

is continuous at $z' = a$ and take the value

$$\bar{g}_2(a) = 60 \cdot \frac{1}{A^4} \sum \frac{1}{m^4}.$$

This means that, in the expansion (30), the terms of negative powers can not occur. Thus, $\bar{g}_2(z')$ is algebraic at $z' = a$. The same may be said at any finite point, if one exists, where $\zeta(z') = 0, 1$, or ∞. Hence $\bar{g}_2(z')$ is algebraic at any finite point. Moreover, the number of values

corresponding to any finite z' can not exceed that of $\zeta(z')$, i.e. a certain integer. Consequently, $\bar{g}_2(z')$ *is a transcendental algebraic function. Similarly for* $\bar{g}_3(z')$ *and* $\gamma(z')\equiv\sqrt[4]{\bar{g}_2(z')/g_2(z)}$.

14. Let there be a finite point, say $z'=a$, such that $\zeta(a)=0$. Then we have, as before (§13), in the vicinity of $z'=a$,

$$\bar{\omega}_1(z')=AF\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta(z)\right)$$

$$\bar{\omega}_2(z')=DF_1\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta(z')\right)$$

$$+\left(C+D\log(\zeta(z'))\right)F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta(z')\right),$$

and $\lim_{z'=a}\bar{g}_2(z')=\frac{60}{A^4}\sum\frac{1}{m^4}$, where the summation is excluded $m=0$. As $\bar{g}_2(z')$ is algebraic in the vicinity of $z'=a$, $\bar{g}_2(z')-\frac{60}{A^4}\sum\frac{1}{m^4}$ has a zero of a finite order (that is, it is not of an infinitesmal order) at $z'=a$. But $\{\log(\zeta(z'))\}^4$ has an infinity of an infinitesimal order at $z'=a$ and

$$\lim_{z'=a}\left\{\log(\zeta(z'))\right\}^4\cdot\left\{\bar{g}_2(z')-\frac{60}{A^4}\sum\frac{1}{m^4}\right\}\neq 0.$$

This is a contradition, and there exists no finite point which satisfies $\zeta(z')=0$. Similarly for $\zeta(z')=1$ and $\zeta(z')=\infty$. Accordingly, $\zeta(z')$ *acquires none of the values* $\zeta=0$, 1 *and* ∞ *in any finite part of the plane.* Consequently $\bar{g}_2(z')$, $\bar{g}_3(z')$ are finite for all finite values of z'.

Similarly, let $\bar{G}_2(z)$, $\bar{G}_3(z)$, $K^2(z)$, $I'(z)$, $\bar{\Omega}_1(z)$, $\bar{\Omega}_2(z)$ be the corresponding functions to $\bar{g}_2(z')$, $\bar{g}_3(z')$, $\varkappa^2(z')$, $\gamma(z')$, $\bar{\omega}_1(z')$, $\bar{\omega}_2(z')$ respectively. Then they have the same properties as the corresponding functions.[1]

15. Suppose that there is a finite point $z'=a$ where $\gamma(a)=0$. Let D be a small region in the z-plane such that all branches of $I'(z)$, $K^2(z)$ and $B(z)$'s are regular in it and that $\lambda\Omega_1(z)+\mu\Omega_2(z)$, ($\lambda$, $\mu=0$, ±1, ±2, ...), are different from those points in a small region containing $z'=a$ in it. Then all the branches of $\wp(z', G_2(z), G_3(z))$ are regular in the field of variation such that z is any point in D, while z' is any point in the vicinity of $z'=a$. Similarly for

[1] See foot note in §10.

$$(27)\quad B_{in_i}(z)\wp^{n_i}(z',\ G_2(z),\ G_3(z))+\cdots\cdots\cdots+B_{i0}(z),\ (i=0,\ 1,\ \cdots\cdots,\ n),$$

and its succesive differential quotients with respect to z', where (27) is the coefficient of $f^i(z, z')$ in (26). As the derivatives of (27) with respect to z' do not all vanish identically for $z'=a$, there is a certain rational number s (zero being included) such that at least one of the limiting values of $(z'-a)^s f(z, z')$ for limit $z'=a$ is algebraic in D. Also, as $\gamma(z')$ and $b(z')$'s are determinate for limit $z'=a$, we have, by

$$\frac{b_{mm_m}(z')\dfrac{1}{\gamma^{2m_m}(z')}\wp^{m_m}\left(\dfrac{z}{\gamma(z')},\ \omega_1(z'),\ \omega_2(z')\right)+\cdots\cdots\cdots+b_{m0}(z')}{(z'-a)^{ms}}$$

$$\times\left\{(z'-a)^s f(z, z')\right\}^m+\cdots\cdots+\frac{b_{0m_0}(z')}{\gamma^{2m_0}(z')}\wp^{m_0}\left(\frac{z}{\gamma(z')},\ \omega_1(z'),\ \omega_2(z')\right)+\cdots\cdots$$

$$+b_{00}(z')=0,\quad (b_{mm_m}(z')\equiv 1),$$

$\lim\limits_{z'=a}\wp\left(\dfrac{z}{\gamma(z')},\ \omega_1(z'),\ \omega_2(z')\right)$ as an algebraic function of z in D. From what has been proved in §§12 and 14, it follows that none of $\bar{\omega}_1(z')$ and $\bar{\omega}_2(z')$ vanishes or becomes infinity for all finite values of z' and that the ratio $\bar{\omega}_2(z')/\bar{\omega}_1(z')$ has always a positive imaginary part. Hence the same is necessarily the case for $\bar{\omega}_1(a)$ and $\bar{\omega}_2(a)$, where $\bar{\omega}_1(a)$ and $\bar{\omega}_2(a)$ are the limiting values of the certain branches of $\bar{\omega}_1(z')$ and $\bar{\omega}_2(z')$ for $z'=a$, which correspond to one and the same branch of $\zeta(z')$. As $\lim\limits_{z'=a}\wp\left(\dfrac{z}{\gamma(z')},\ \bar{\omega}_1(z'),\ \bar{\omega}(z')\right)$ is algebraic, we may suppose that those which correspond to the above assigned branch of $\zeta(z')$ are $\wp(a_i(z, a), \bar{\omega}_1(a), \bar{\omega}_2(a))$, $(i=1,\ \cdots\cdots,\ n)$, where, considering that z is an arbitrarily assigned value, $a_i(z, a)$ is a point in the fundamental parallelogram. Therefore $a_i(z, a)\not\equiv a_j(z, a)\pmod{\bar{\omega}_1(a), \bar{\omega}_2(a)}$ for $i\neq j$, $(i, j=1, 2, \cdots\cdots, n)$. For any prescribed positive value ε, there is a corresponding positive value σ such that

$$\frac{z}{\gamma(z')}=p\bar{\omega}_1(z')+q\bar{\omega}_2(z')+a_i(z, z_2')$$

and

$$|\bar{\omega}_1(z')-\bar{\omega}_1(a)|<\sigma$$

$$|\bar{\omega}_2(z')-\bar{\omega}_2(a)|<\sigma$$

$$|a_i(z, z')-a_i(z, a)|<\sigma$$

for all values of z', which satisfy $0<|z'-a|<\varepsilon$, where $a_i(z, a)$ is certain one of $a_1(z, a), a_2(z, a), \ldots\ldots, a_n(z, a)$ and p, q are certain integers. Determine ε so small that 2σ is less than all $|a_i(z, a)-a_j(z, a)|$, $(i, j=1, \ldots\ldots, n;\ i\gtrless j)$. Suppose $z'=z_1'$, $z'=z_2'$ be two distinct points in the region $0<|z'-a|<\varepsilon$, such that in the equations

$$\frac{z}{\gamma(z_1')}=p_1\bar{\omega}_1(z_1')+q_1\bar{\omega}_2(z')+a_{i_1}(z,\ z_1')$$

and

$$\frac{z}{\gamma(z_2')}=p_2\bar{\omega}_1(z_1')+q_2\bar{\omega}_2(z')+a_{i_2}(z,\ z_2'),$$

at least one of the inequalities $p_1\gtrless p_2$, $q_1\gtrless q_2$, $i_1\gtrless i_2$ holds. Then as $\frac{z}{\gamma(z')}$ takes every value in the vicinity of infinity, there is at least one point $z'=z_0'$ on the straight line joining $z'=z_1'$ and $z'=z_2'$, for which

$$\frac{z}{\gamma(z_0')}=p_1'\bar{\omega}_1(z_0')+q_1'\bar{\omega}_2(z_0')+a_{j_1}(z,\ z_0')=p_2'\bar{\omega}_1(z_0')+q_2'\bar{\omega}_2(z_0')+a_{j_2}(z,\ z_0'),$$

where at least one of the inequalities $p_1'\gtrless p_2'$, $q_1'\gtrless q_2'$, $j_1\gtrless j_2$ holds. Hence we have

$$(31)\quad p\bar{\omega}_1(z_0')+q\bar{\omega}_2(z_0')=a_{j_1}(z,\ z_1')-a_{j_2}(z,\ z_0'),$$

where, in virtue of $\tau(z_0')=\frac{\bar{\omega}_2(z_0')}{\bar{\omega}_1(z_0')}\gtrless$real, $a_{j_1}(z,\ z_0')\gtrless a_{j_2}(z,\ z_0')$ or $j_1\gtrless j_2$. There exist infinitely many points such as $z'=z_0'$ for which the relation (31) holds. But the number of all possible combinations of 1, 2, ..., n, such as (j_1, j_2), $(j_1\gtrless j_2)$ is ${}_nC_2=\frac{n(n-1)}{2}$ and accordingly there is at least one combination, say (j_1, j_2), for which the relation (31) holds for infinitely many values of z', which converge to $z'=a$. In (31), consider the limit $z'=a$. The right number converges to a finite value $a_{j_1}(z, a)-a_{j_2}(z, a)$. Therefore, as $\frac{\bar{\omega}_2(a)}{\bar{\omega}_1(a)}$ has a positive imaginary part, none of p and q becomes infinity for limit $z'=a$, and accordingly, there are infinitely many values of z' which converge to $z'=a$ and for which (31) holds for the same values of p, q, j_1 ann j_2. In the limit $z'=a$, we have

$$p\bar{\omega}_1(a)+q\bar{\omega}_2(a)=a_{j_1}(z, a)-a_{j_2}(z, a),\quad (j_1\gtrless j_2),$$

i.e.

$$a_{j_1}(z, a)=a_{j_2}(z, a)\quad (\mathrm{mod}\ \bar{\omega}_1(a),\ \bar{\omega}_2(a)),\ (j_1\gtrless j_2),$$

which is a contradiction. Similarly for all branches of $\zeta(z')$. Hence, $\gamma(z')$ *never acquires the value zero*

16. Let G be the aggregate of all branch points of at least one of $\gamma(z')$ and $\zeta(z')$. Then there are four cases to be considered.

i) G contains no point.
ii) G contains at least two points which are not branch points of $f(z, z')$ where z is an arbtrary constant.
iii) all points in G, with one exception at most, are branch points of $f(z, z')$ for any assigned value of z, and are infinite in number.
iv) all points in G, with one exception at most, are branch points of $f(z, z')$ for any assigned value of z, but are finite in number.

We now investigate these cases successively.

i) As G contains no branch point of $\gamma(z')$ and of $\zeta(z')$, they are mermorphic functions of z'. $\zeta(z')$ acquires, however, none of the values $\zeta=0$, 1 and ∞ for finite values of z', and hence by Picard's theorem,[1] it is a constant. As $\gamma(z')$ has no zero point at finiteness, $\frac{1}{\gamma(z')}$ is an entire function. As $\zeta(z')$ is a constant, so is $\bar{g}_2(z')$. Therefore $\frac{1}{\gamma(z')} \equiv \sqrt[4]{\frac{g_2(z')}{\bar{g}_2(z')}}$ is, as $g_2(z')$, an entire function, at most, of the 2^{nd} order. We now substitute $\frac{1}{\gamma^2(z')}\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ for $\wp(z, \gamma(z)\bar{\omega}_1, \gamma(z')\bar{\omega}_2)$ in (25). Then, as $b(z')$'s are transcendental algebraic functions, at most of the 2^{nd} order (§10) and $f(z, z')$ is, by (26), a transcendental algebraic function of the 2^{nd} order in z' for any assigned value of z, $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ is, by (25), a transcendental algebraic function, at most, of the 2^{nd} order in z' for any assigned value of $z(\neq 0)$. $\frac{1}{\gamma(z')}$ being, however, an entire function, $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ is a mermorphic function at most of the 2^{nd} order in z' for any assigned value of $z(\neq 0)$. But $\wp\left(\frac{\varepsilon}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$, $(z \neq 0)$, is of the 2^{nd} order in $\frac{1}{\gamma(z')}$. Accordingly for any prescribed positive value ε, there is a corresponding vulue R such that

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[1] Borel: *Leçons sur les fonctions mermorphes*, p. 66.

$$\left|\frac{1}{\gamma(z')}\right| \leqq |z'|^{1+\varepsilon} \qquad \text{for all} \qquad |z'| \geqq R.$$

Hence, $\frac{1}{\gamma(z')}$ is a polynomial of the form $\alpha z'+\beta$, where α and β are constants. We conclude, therefore, that *if both* $\gamma(z')$ *and* $\zeta(z')$ *have no branch point, then* $\zeta(z')$ *is a constant and*

$$\gamma(z') = \frac{1}{\alpha z'+\beta}.$$

17. ii) As G contains at least two points which are not branch points of $f(z, z')$, let one of them be $z'=a$. Suppose that $\zeta^{(1)}(z')$, $u_1^{(1)}(z')$, $u_2^{(1)}(z')$, $\bar{\omega}_1^{(1)}(z')$, $\bar{\omega}_2^{(1)}(z')$, $\gamma^{(1)}(z')$ become $\zeta^{(2)}(z')$, $u_1^{(2)}(z')$, $u_2^{(2)}(z')$, $\bar{\omega}_1^{(2)}(z')$, $\bar{\omega}_2^{(2)}(z')$, $\gamma^{(2)}(z')$ respectively after a single description of a closed circuit round the branch point $z'=a$. Then, we have two equations of the form (25), which correspond to $\zeta^{(1)}(z')$ and $\zeta^{(2)}(z')$ respectively. As $z'=a$ is, by assumption, not a branch point of $f(z, z')$, $f(z, z')$ has the same values in both equations. Eliminating $f(z, z')$ from them, we have an algebraic relation between $\wp(z, \omega_1^{(1)}(z'), \omega_2^{(1)}(z'))$ and $\wp(z, \omega_1^{(2)}(z'), \omega_2^{(2)}(z'))$, and accordingly $\omega_1^{(1)}$, $\omega_2^{(1)}$, $\omega_1^{(2)}$, $\omega_2^{(2)}$ are connected by

$$(32) \quad \begin{cases} p_1\omega_1^{(2)} = q_1\omega_1^{(1)} + r_1\omega_2^{(1)} \\ p_2\omega_2^{(2)} = q_2\omega_1^{(1)} + r_2\omega_2^{(1)} \end{cases}$$

where p_i, q_i, r_i, $(i=1, 2)$, are certain integers and p_1, p_2 are different from zero.

First, we assnme that $\gamma(z')$ has no infinity point at finiteness. As $\bar{\omega}_1(z')$ and $\bar{\omega}_2(z')$ are always finite and different from zero at finiteness (§§ 12 and 14), $\omega_1(z')$ and $\omega_2(z')$ vanish or become infinity simultaneously according as $\gamma(z')$ vanishes or become infinity. $\gamma(z')$ has, however, no zero point (§15) and no infinity point (by assumption), so that we have, at $z'=a$, $\omega_1^{(1)}(a)=\omega_1^{(2)}(a)$ and $\omega_2^{(1)}(a)=\omega_2^{(2)}(a)$, which are both finite. Accordingly, we have, in (32), $r_1=q_2=0$. For, if $r_1 \neq 0$, then $\tau(a)=\frac{p_1-q_1}{r_1}$, i.e. $\tau(a)$ is a real quantity, which is evidently impossible. Similarly for q_2. Consequently, $p_1=q_1$ and $p_2=r_2$, and we have, therefore,

$$\gamma_1^{(1)}(z')\bar{\omega}_1^{(1)}(z') = \gamma^{(2)}(z')\bar{\omega}_1^{(2)}(z')$$

$$\gamma_1^{(1)}(z')\bar{\omega}_2^{(1)}(z') = \gamma^{(2)}(z')\bar{\omega}_2^{(2)}(z'),$$

or

$$\frac{\bar{\omega}_2^{(1)}(z')}{\bar{\omega}_1^{(1)}(z')}=\frac{\bar{\omega}_2^{(2)}(z')}{\bar{\omega}_1^{(2)}(z')}.$$

We have, however, by (29)

$$\bar{\omega}_1(z')=Au_1\{\zeta(z')\}+Bu_2\{\zeta(z')\}$$
$$\bar{\omega}_2(z')=Cu_1\{\zeta(z')\}+Du_2\{\zeta(z')\},$$

where A, B, C, D are constants such that the imaginary part of $\frac{Cu_1+Du_2}{Au_1+Bu_2}$ is always positive and $AD-BC\neq 0$.

Hence.

$$\frac{Cu_1^{(1)}(z')+Du_2^{(1)}(z')}{Au_1^{(1)}(z')+Bu_2^{(1)}(z')}=\frac{Cu_1^{(2)}(z')+Du_2^{(2)}(z')}{Au_1^{(2)}(z')+Bu_2^{(2)}(z')},$$

or

$$\frac{u_2^{(1)}(z')}{u_1^{(1)}(z')}=\frac{u_2^{(2)}(z')}{u_1^{(2)}(z')},$$

or precisely,

$$(33)\quad u_2\{\zeta^{(1)}(z)\}u_1\{\zeta^{(2)}(z')\}=u_2\{\zeta^{(2)}(z')\}u_1\{\zeta^{(1)}(z')\}.$$

Suppose that $(\lambda-1)$ be the multiplicity of the branch point $z'=a$.

$$(34)\quad\begin{cases}\zeta(z')=a_0+a_1(z'-a)^{\frac{1}{\lambda}}+\cdots\cdots\cdots+a_i(z'-a)^{\frac{i}{\lambda}}+\cdots\cdots\cdots, & (a_0\neq 0)\\ u_1(\zeta)=a_0+a_1(\zeta-a_0)+\cdots\cdots\cdots+a_i(\zeta-a_0)^i+\cdots\cdots\cdots, & (a_0\neq 0)\\ u_2(\zeta)=\quad b_1(\zeta-a_0)\quad+\cdots\cdots\cdots+b_i(\zeta-a_0)^i+\cdots\cdots\cdots, & (b_1\neq 0).\end{cases}$$

Let $\sigma=\cos\frac{2\pi}{\lambda}+\sqrt{-1}\sin\frac{2\pi}{\lambda}$ and μ be the least integer such that $a_\mu\neq 0$ and $\mu\not\equiv 0\pmod{\lambda}$. Substituting (34) in (33), we have

$$\left[b_1\left\{a_1(z'-a)^{\frac{1}{\lambda}}+\cdots\cdots+a_i(z'-a)^{\frac{i}{\lambda}}+\cdots\cdots\right\}+\cdots\cdots\right]$$
$$\times\left[a_0+a_1\left\{a_1\sigma(z'-a)^{\frac{1}{\lambda}}+\cdots\cdots+a_i\sigma^i(z'-a)^{\frac{i}{\lambda}}+\cdots\cdots\right\}+\cdots\cdots\right]$$
$$=\left[b_1\left\{a_1\sigma(z'-a)^{\frac{1}{\lambda}}+\cdots\cdots+a_i\sigma^i(z'-a)^{\frac{i}{\lambda}}+\cdots\cdots\right\}+\cdots\cdots\right]$$
$$\times\left[a_0+a_1\left\{a_1(z'-a)^{\frac{1}{\lambda}}+\cdots\cdots+a_i(z'-a)^{\frac{i}{\lambda}}+\cdots\cdots\right\}+\cdots\cdots\right].$$

Comparing the coefficient of $(z'-a)^{\frac{\mu}{\lambda}}$, we have

$$a_0 b_1 \alpha_\mu = a_0 b_1 \alpha_\mu \sigma^\mu,$$

which is impossible unless $\sigma^\mu = 1$. Consequently $\zeta(z')$ can not have any branch point at finiteness, so that it is a constant as before (§16). Accordingly $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{g}_2$, $\bar{g}_3$ are all constants.

Similarly we may prove that $z'=a$ can not be a branch point of $\gamma(z')$, and accordingly that it can not be a point of G. We conclude, therefore, that *any point of G, which is not a branch point of $f(z, z)$ for any constant value of z, must be an infinity point of $\gamma(z')$.*

18. Let $z'=a$ be any one of such points and transfer the origin to that point. In virtue of (1), we may easily prove that, for suitably assigned value of $z_1, z_2, \ldots\ldots, z_{i-1}, z_{i+1}, \ldots\ldots, z_n$, $f((z_i))$ is a transcendental algebraical function of z_i. Hence, in the vicinity of the new origin, each branch of $f(z, z'+a)$ is expensible as follows.

$$(24)' \quad f(z, z'+a) = \overline{A}_{-\mu}(z)\, z'^{-\frac{\mu}{\lambda}} + \ldots\ldots + \overline{A}_0(z) + \overline{A}_1(z)\, z'^{\frac{1}{\lambda}} + \ldots\ldots\ldots + \overline{A}_i(z)\, z'^{\frac{i}{\lambda}} + \ldots\ldots\ldots,$$

where $\lambda(>0)$ and $\mu(\geqq 0)$ are integers. We have also

$$(26)' \quad \{\overline{B}_{n'n'_{n'}}(z)\wp^{n'_{n'}}(z', \Omega_1(z), \Omega_2(z)) + \ldots\ldots + \overline{B}_{n'0}(z)\} f^{n'}(z, z'+a) + \ldots\ldots + \{\overline{B}_{0n'_0}(z)\wp^{n'_0}(z', \Omega_1(z), \Omega_2(z)) + \ldots\ldots + \overline{B}_{00}(z)\} = 0, (\overline{B}_{n'n'_{n'}}(z) \equiv 1),$$

and

$$(25)' \quad \{b_{mm_m}(z'+a)\wp^{m_m}(z, \omega_1(z'+a), \omega_2(z'+a)) + \ldots\ldots + b_{m0}(z'+a)\} \times f^m(z, z'+a) + \ldots\ldots + \{b_{0m_0}(z'+a)\wp^{m_0}(z, \omega_1(z'+a), \omega_2(z'+a)) + \ldots\ldots + b_{00}(z'+a)\} = 0, \ (b_{mm_m}(z'+a) \equiv 1).$$

As $g_2(z'+a)$, $g_3(z'+a)$ and $b(z'+a)$'s are algebraic in the vicinity of the new origin (§10), we may determine a positive integer ν, ($\nu \equiv 0 \pmod{\lambda}$) such that all the branches of $g_2(z'+a)$, $g_3(z'+a)$ and $b(z'+a)$'s are uniform functions of $x \equiv z'^{\frac{1}{\nu}}$ in the assigned region, having the new origin as a regular of a non-essential singular point. Hence all branches of $z^2\wp(z, g_2(z'+a), g_3(z'+a))$ are regular functions of z and x in that

region and are successively differentiable with respect to x; and so also for all the coefficients of the powers of $f(z, z'+a)$ in (25)′. Supposing that a be the order of infinity of $\gamma(z'+a)$ at $z'=0$, the coefficient of z^{2i}, $(i=1, 2, \ldots\ldots)$, in the expansion of $\wp(z, g_2(z'+a), g_3(z'+a))$ in power series of z has a zero at least of the $(2i+2)a^{th}$ order at $z'=0$. Accordingly, all the derivatives of $\wp(z, g_2(z'+a), g_3(z'+a))$ with respect to x are polynomials of z for $x=0$; and so also for all the coefficients of the powers of $f(z, z'+a)$ in (25)′. Substitute the fractional power series in (24)′ for $f(z, z'+a)$ in (25)′. Then, differentiating successively both the members with respect to x and assigning zero for x, we have $\overline{A}_i(z)$, $(i=-\mu, \ldots\ldots, 0, 1, 2, \ldots\ldots)$ as an algebraic function of z. Accordingly $G_2(z)$, $G_3(z)$, $K^2(z)$ and $\overline{B}(z)$'s are all algebraic functions (§9).

We proved that in the region where $\zeta(z')$ is algebraic—even if $\zeta(z')$ acquires some of the values 0, 1 and ∞—$\bar{g}(z')$ is also algebraic (§13), and that in the region where $\bar{g}(z')$ is algebraic, $\zeta(z')$ can not acquire any one of the values 0, 1 and ∞ (§14). By the same reasoning, $K^2(z)$ can not acquire any one of the values 0, 1 and ∞ in the whole Gauss plane (including the point at infinity), which is impossible unless $K^2(z)$ is a constant. Accordingly $\overline{G}_2(z)$ and $\overline{G}_3(z)$ are constants and $\Gamma(z)\equiv\sqrt[4]{\overline{G}_2(z)/G_2(z)}$ is an ordinary algebraic function.

As there are at least two points in G, which are not branch points of $f(z, z'+a)$ for any assigned values of z, let $z'=b(\neq 0)$ be any one of them different from the new origin. Then it must be, by §17, an infinity point of $\gamma(z'+a)$ and accordingly $f(z, a+b)$ is, by (25)′, an ordinary algebraic function of z. But $\Gamma(z)$ and $B(z)$'s are also. Consequently $\wp\left(\frac{b}{\Gamma(z)}, \overline{\Omega}, \overline{\Omega}_2\right)$ is, by (26)′, an ordinary algebraic function of z. This is evidently impossible, and hence *the case* ii) *can never occur.*

19. iii) In this case, G contains infinitely many points which are branch points of $f(z, z')$ for any assigned value of z. Let G' be the aggregate of those points. As all independent variables except $z_2\equiv z'$ are considered as parameters, (1) reduces to the form

$$F_1(\varphi(z_1'),\ \varphi(z_2'),\ z_1'+z_2')=0,$$

where $\varphi(z')\equiv f(z, z')$ and $F_1(u, v, w)$ is an irreducible polynominal of u and v, and is analytic with respect to w in the whole Gauss plane (§8). Considering $z_1'+z_2'=c$ be a constant, let the discriminant of F_1

with respect to $\varphi(z_1')$ be $F_2(\varphi(z_2'), c)$. Then $F_2=0$ at any point in G'' where G'' is an aggregate of infinitely many points, each point z_2' of which corresponds to a certain point z' of G' by the relation $z_1'+z_2'=c$ and inversely. As F_2 is a polynominal of $\varphi_2(z_2')$, we may easily prove that all the points in G'' may be divided into a finite number of systems such that any two points in one and the same system are congruent (mod $\Omega_1(z)$, $\Omega_2(z)$), while any two points in any two different systems are incongruent (mod $\Omega_1(z)$, $\Omega_2(z)$). As G'' contains infinitely many points, there is at least one system which contains infinitely many points. This is, however, impossible unless there exist two integers p, q($p=q=0$ being excluded) such that $p\,\Omega_1(z)+q\Omega_2(z)=$constant. Put $p\Omega_1(z)+q\Omega_2(z)\equiv\Omega_1'$ and determine $\Omega_2'(z)$ such that $\Omega_2'(z')\equiv p'\Omega_1(z)+q'\Omega_2(z)$ and $\dfrac{\Omega_2'(z)}{\Omega_1'(z)}$ has always a positive imaginary part, where p', q' are integers which are relative prime and satisfy $pq'-p'q=1$. Then we may take Ω_1' and $\Omega_2'(z)$ instead of $\Omega_1(z)$ and $\Omega_2(z)$ in (26) without effecting any change on $G_2(z)$, $G_3(z)$, $\Gamma(z)$ and $B(z)$'s. If there be any system which contains infinitely many incongruent points (mod Ω_1'), we have also $\Omega_2'(z)=$constant. Hence both Ω_1' and $\Omega_2'(z)$ are constants, i.e. both $\Omega_1(z)$ and $\Omega_2(z)$ are independent of z, which is against the assumption (§9). Accordingly all points in G'' may be divided into a finite number of systems of congruent points (mod Ω_1').

Considering z as a parameter, $f(z, z')$ has a constant period Ω_1': and so has $a_i(z')$, ($i=0, 1, 2, \ldots\ldots$), in (23). Accordingly so also for $g_2(z')$, $g_3(z')$, $\gamma(z')$, $\zeta(z')$ and $b(z')$'s in (24). Put $e^{\frac{2\pi\sqrt{-1}}{\Omega_1'}z'}\equiv x$. Then these functions are transcendental algebraic functions of x. From what has been proved, it follows that ζ has, as a function of x, at most only a finite number of branch points, and accordingly it reduces, by theorem 8 in Chap. II.,[1] to an ordinary algebraic function of x. By the same reasoning as in §18, $\zeta(z')$ reduces to a constant and accordingly $\bar{g}_2(z')$, $\bar{g}_3(z')$ are also constants. Similarly as in §16, we have, for each branch of $\gamma(z')$,

$$\left|\frac{1}{\gamma(z')}\right| \leqq |z'|^{1+\varepsilon} \qquad \text{for all} \quad |z'| \geqq R,$$

where ε is any prescribed positive value and R is a corresponding value

[1] My first paper; these Memoirs, vol. VI., no. 3.

to ε. Accordingly $\gamma(z')$ must be an ordinary algebraic function, which has a period Ω_1'. This is evidently impossible and we conclude that *case* iii) *can never occur.*

20. iv). In this case, G contains only a finite number of points, so that the number of branch points of $\zeta(z')$ is finite. Accordingly, by the same reasoning as in §19, $\zeta(z')$, $\bar{g}_2(z')$, $\bar{g}_3(z')$ are constants, and $\gamma(z')$ is an ordinary algebraic function. Also $\gamma(z')$ has no zero point at finiteness (§15) and may be, therefore, expressed by an irreducible equation

$$(35)\quad T_0(z')\gamma^t + T_1(z')\gamma^{t-1} + \cdots\cdots + 1 = 0,$$

where t is a positive integer and $T_0(z')$, $T_1(z')$, ..., $T_{t-1}(z')$ are polynomials of z.

Similarly, the foregoing reasoning (§§15–20) holds for $\Gamma(z)$ and $K^2(z)$, and we may conclude that $K^2(z)$ is a constant while $\Gamma(z)$ is an ordinary algebraic function which has no zero point at finiteness.

All the derivatives of $\wp(z',\ \Omega_1(z),\ \Omega_2(z)) = \frac{1}{\Gamma(z)^2}\wp\left(\frac{z'}{\Gamma(z)},\ \overline{\Omega}_1,\ \overline{\Omega}_2\right)$, $(z' \neq 0)$, with respect to z are algebraic functions of z' and $\wp\left(\frac{z'}{\Gamma(z)},\ \overline{\Omega}_1,\ \overline{\Omega}_2\right)$ where z is considered as a parameter after differentiation. Similarly for

$$(27)\quad B_{in_i}(z)\wp^{n_i}(z', G_2(z), G_3(z)) + \cdots\cdots + B_{i0}(z),\ (i = 0, 1, \ldots\ldots, n),$$

which is the coefficient of $f^i(z, z')$, in (26). Substitute the power series (23) instead of $f(z, z')$ in (26). Then differentiating with respect to z and assigning the value zero for z, we have $a_i(z')$, $(i = 0, 1, 2, \ldots\ldots)$, as algebraic functions of z and $\wp(z', \Omega_1(0), \Omega_2(0))$. Similarly for $\gamma(z')$ and $b(z')$'s $(b_{mm_m}(z') \equiv 1)$. By eliminating $f(z, z')$ from (25) and (26), we have

$$\Phi\left(z', \wp(z', \Omega_1(0), \Omega_2(0)), \wp(z', \Omega_1(z), \Omega_2(z)), \wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)\right) = 0,$$

where Φ is a polynomial of z', $\wp(z', \Omega_1(0), \Omega_2(0))$, $\wp(z', \Omega_1(z), \Omega_2(z))$ and $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ whose coefficients are functions of z. Φ contains $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ explicitly. For otherwise, at least one of the values of $f(z, z')$ which are determined by (26) would satisfy (25) for any value

of $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$, and accordingly there would be an algebraic relation between $\wp(z, \gamma(z')\bar{\omega}_1, \gamma(z')\bar{\omega}_2)$ and $\wp(z, \gamma(z')\bar{\omega}_1^{(1)}, \gamma(z')\bar{\omega}_2^{(1)})$ for any values of z and z', where $\bar{\omega}_1^{(1)}$, $\bar{\omega}_2^{(1)}$ are arbitrary constants such that $\frac{\bar{\omega}_2^{(1)}}{\bar{\omega}_1^{(1)}}$ has a positive imaginary part. But this is evidently impossible. There are infinitely many values of z whose limiting value is zero and for which $I(z)/I(0)$ is a rational number. Let $z=z_0$ be any one of them. Then, we may assume that the function Φ for $z=z_0$ contains $\wp\left(\frac{z_0}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ explicitly. For otherwise, Φ does not contain $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ explicitly for infinitely many values of z whose limiting value is zero: and so also for all values of z in the vicinity of $z=o$, which is impossible. For $z=z_0$, $\Omega_1(0)$, $\Omega_2(0)$, $\Omega_1(z_0)$, $\Omega_2(z_0)$ are connected by the relations (32) and hence between $\wp(z', \Omega_1(0), \Omega_2(0))$ and $\wp(z', \Omega_1(z_0), \Omega_2(z_0))$, there exists an algebraic relation. Accordingly $\Phi=0$ reduces to

$$\Phi_1\left(z', \wp(z', \Omega_1(0), \Omega_2(0)), \wp\left(\frac{z_0}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)\right)=0.$$

As $\wp\left(\frac{z_0}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ is not an ordinary algebraic function, Φ_1 contains $\wp(z', \Omega_1(0), \Omega_2(0))$ explicitly. Accordingly $\wp\left(\frac{z_0}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ must be a transcendental algebraic function of the 2^{nd} order in z', so that for each branch of $\gamma(z')$, there exist infinitely many values of z', say z_i', $(i=1, 2, \ldots\ldots; \lim_{i=\infty} z_i'=\infty)$, for which $\lim_{i=\infty}\frac{\gamma(z_i')}{z_i'}=1$. Therefore, $T_0(z')$ is a polynomial of the t^{th} degree, and $\gamma(z')$ has at least one infinitely point. Let $z'=a$ be one of them, and transfer the origin to that point. Then we have, as before (§18), (25)′ and (26)′ where $\overline{B}(z)$'s, $(\overline{B}_{nn_n}(z)\equiv 1)$, are algebraic functions of z. If G contains at least one point, say $z'=b$, which is different from the new origin and is a branch point of $f(z, z'+a)$ for any assigned value of z, the discriminant $\mathfrak{D}(\overline{B}(z), \wp(z', \Omega_1(z), \Omega_2(z))$ of (26)′ will vanish identically for $z'=b$ and for all values of z, where $\mathfrak{D}$ is a polynomial of $\wp(z', \Omega_1(z), \Omega_2(z))$ and $\overline{B}(z)$'s. Accordingly, $[\mathfrak{D}]_{z'=b}=0$ shows that $\wp(b, \Omega_1(z), \Omega_2(z))$ or $\wp\left(\frac{b}{I'(z)}, \overline{\Omega}_1, \overline{\Omega}_2\right)$ is an algebraic function of z, which is impossible. Next, if there be a point of

G, which is different from the new origin and is not a branch point of $f(z, z'+a)$, then it must be an infinity point of $\gamma(z'+a)$ (§17). But similarly, as in §18, we may prove that $\gamma(z'+a)$ has at most only one infinity point at finiteness. Consequently in all cases, $\gamma(z'+a)$ has no branch point, except the new origin at most, that is to say, $\frac{1}{\gamma(z'+a)}$ has at most only one singular point—the branch point at the new origin —in the finite part of the z'-plane. Hence, in the vicinity of the new origin, all the branches of $\frac{1}{\gamma(z'+a)}$ may be expressed in the form

$$\frac{1}{\gamma(z'+a)}=y^s(a_0+a_1y+a_2y^2+\ldots\ldots+a_iy^i+\ldots\ldots),\ (a_0\neq 0),$$

where y is a t^{th} root of z' and s is a certain positive integer. Then the above expression is regular for all finite values of y, so that it is a polynomial or an entire function of y. As $\frac{1}{\gamma(z'+a)}$ is, however, an algebraic function of z', it can not be an entire function of $y\equiv z'^{\frac{1}{t}}$ i. e. it is a polynomial of y. Moreover, as $\frac{1}{\gamma(z'+a)}$ has no zero point except the new origin at finiteness, we have

$$\frac{1}{\gamma(z'+a)}=a_0y^s=a_0z'^{\frac{s}{t}}.$$

As $\lim_{i=\infty}\frac{\gamma(z_i')}{z_i'}=1$ referring to the old origin, we have $s=t$, i.e. $\frac{1}{\gamma(z'+a)}=a_0z'$. Moreover, we have $\zeta(z')=$constant, and hence we conclude that *case* iv) *can never occur.*

21. In short, all the cases except the first one are inadmissible, and we have necessarily $\gamma(z')=\frac{1}{a'z'+\beta'}$ and $\zeta(z')$ constant. Similarly by repeating the same reasoning, we have $\Gamma(z)=\frac{1}{az+\beta}$ and $K^2(z)=$ constant.

Now we transfer the origin to the point $\left(\frac{-\beta}{a}, \frac{-\beta'}{a'}\right)$ and (25) becomes

$$(25)'' \quad \left\{\bar{b}_{m'm'_1m'_1}(z')\wp^{m'm'}(z')\wp\left(z, \frac{\bar{\omega}_1}{\alpha' z'}, \frac{\bar{\omega}_2}{\alpha' z'}\right)+\cdots\cdots+\bar{b}_{m'0}(z')\right\}$$

$$\times f^{m'}\left(z-\frac{\beta}{\alpha}, z'-\frac{\beta'}{\alpha'}\right)+\cdots\cdots$$

$$+\left\{\bar{b}_{0m'_0}(z')\wp^{m'0}\left(z, \frac{\bar{\omega}_1}{\alpha' z'}, \frac{\bar{\omega}_2}{\alpha' z'}\right)+\cdots\cdots+\bar{b}_{00}(z')\right)=0,$$

$$\left(\bar{b}_{m'm'_1m'_1}(z')\equiv 1\right),$$

where $\bar{b}$'s are algebraic functions of z' (§18). The equation $(25)''$ may, therefore, by written in the form

$$(36) \quad P_0\left(\wp(zz', \frac{\bar{\omega}_1}{\alpha'}, \frac{\bar{\omega}_2}{\alpha'}), f\left(z-\frac{\beta}{\alpha}, z'-\frac{\beta'}{\alpha'}\right)\right)z'^{l}+\cdots\cdots$$

$$+P_l\left(\wp\left(zz'+\frac{\bar{\omega}_1}{\alpha'}, \frac{\bar{\omega}_1}{\alpha'}\right), f\left(z-\frac{\beta}{\alpha}, z'-\frac{\beta'}{\alpha'}\right)\right)=0,$$

where $P_0, P_1, \cdots\cdots, P_l$ are polynomials of $\wp$ and f. We may determine the field of variation $E(z, z')$ of z and z' such that for any values of z and z' in $E(z, z')$, f and P's are continuous. First, we assume that there exist three integers p, q, $r(p\neq 0)$, which satisfy the relation.

$$(37) \quad p\frac{\Omega_1}{\alpha}=q\frac{\bar{\omega}_1}{\alpha'}+r\frac{\bar{\omega}_2}{\alpha'}.$$

Substitute $z'+p\frac{\Omega_1}{\alpha z}i$, $(i=0, 1, 2, \cdots\cdots)$, for z' in (36). Then, as

$$f\left(z-\frac{\beta}{\alpha}, z'-\frac{\beta'}{\alpha'}+\frac{\Omega_1}{\alpha z}pi\right)=f\left(z-\frac{\beta}{\alpha}, z'-\frac{\beta'}{\alpha'}\right) \quad ((26)) \quad \text{and}$$

$$\wp\left(z\left(z'+\frac{\Omega_1}{\alpha z}pi\right)+\frac{\bar{\omega}_1}{\alpha'}, \frac{\bar{\omega}_2}{\alpha'}\right)=\wp\left(zz'+\frac{\bar{\omega}_1}{\alpha'}qi+\frac{\bar{\omega}_2}{\alpha'}ri, \frac{\bar{\omega}_1}{\alpha'}, \frac{\bar{\omega}_2}{\alpha'}\right)$$

$$=\wp\left(zz', \frac{\bar{\omega}_1}{\alpha'}, \frac{\bar{\omega}_2}{\alpha'}\right),$$

we have $P_0(\wp, f)\left(z'+p'\frac{\overline{\Omega}_1}{\alpha z}i\right)^l+\ldots\ldots\ldots+P_l(\wp, f)=0.$

Considering limit $i=\infty$, we have necessarily

$$P_0\left(\wp\left(zz', \frac{\overline{\omega}_1}{\alpha'}, \frac{\overline{\omega}_2}{\alpha'}\right), f\left(z-\frac{\beta}{\alpha}, z'-\frac{\beta'}{\alpha'}\right)\right)\equiv 0.$$

Secondly we suppose that there are no integers which satisfy (37). Then, we may find integers p_i, q_i, r_i, $(i=1, 2, \ldots\ldots\ldots)$, such that

$$p_i\frac{\overline{\Omega}_1}{\alpha}=q_i\frac{\overline{\omega}_1}{\alpha'}+r_i\frac{\overline{\omega}_2}{\alpha'}+\varepsilon_i$$

where $\lim_{i=\infty}\varepsilon_i=0$. For z, z' in $E(z, z')$, we have

$$\lim_{i=\infty}f\left(z-\frac{\beta}{\alpha}, z'-\frac{\beta'}{\alpha'}+p_i\frac{\overline{\Omega}_1}{\alpha}-\varepsilon_i\right)=\lim_{i=\infty}f\left(z-\frac{\beta}{\alpha}, z'-\frac{\beta'}{\alpha'}-\varepsilon_i\right)$$

$$=f\left(z-\frac{\beta}{\alpha}, z'-\frac{\beta'}{\alpha'}\right).$$

Accordingly, substituting $z'+\frac{p_i\overline{\Omega}_1-\alpha\varepsilon_i}{\alpha z}=z'+\frac{q_i\overline{\omega}_1}{\alpha' z}+\frac{r_i\overline{\omega}_2}{\alpha' z}$ for z in (36), we have similarly

$$P_0\left(\wp\left(zz', \frac{\overline{\omega}_1}{\alpha'}, \frac{\overline{\omega}_2}{\alpha'}\right), f\left(z-\frac{\beta}{\alpha}, z'-\frac{\beta'}{\alpha'}\right)\right)\equiv 0.$$

Hence, in both cases, $f\left(z-\frac{\beta}{\alpha}, z'-\frac{\beta'}{\alpha'}\right)$ is an algebraic function of $\wp(zz')$ where the periods as well as the coefficients are independent of z and z'. If $\overline{\omega}_1$ or $\overline{\omega}_2$ be a function of some of z_i $(i=3, 4, \ldots\ldots\ldots, n)$, then, we repeat the same reasoning as before and accordingly, referring to the initial origin, we have generally

$$P(\wp((z_1+\alpha_1)(z_2+\alpha_2)\ldots\ldots\ldots(z_r+\alpha_r), \lambda, \mu), f((z_i)))=0,$$

where the periods λ, μ and α_1, α_2, ..., α_r are absolute constants and the coefficients are independent of z_1, z_2, ..., z_r.

22. Consider z_{ij}, $(i=r+1, \ldots\ldots, n;\ j=1, 2)$, as parameters and put $z_{i1}=z_{i2}=z_i$, $(i=1, 2, \ldots\ldots, r)$. Then we have $f_{i1}=f_{i2}=f_i$, $(i=1, 2, \ldots\ldots, r)$, and (1) becomes

$$(1)'\quad F(f_1, f_2, \ldots\ldots, f_r, f_{r+11}, f_{r+12}, \ldots\ldots, f_{n2}, f_0)=0.$$

Supposing that $a_2\neq 0$, we assign suitably chosen values to z_i, $(i=2, 3, \ldots\ldots, r)$, such that $\frac{2z_i+a_i}{z_i+a_i}$, $(i=3, \ldots\ldots, r)$, is rational while $\frac{2z_2+a_2}{z_2+a_2}$ is real and irrational. In $(1)'$, f_1 is a doubly periodic function of z_1 whose periods are

$$\frac{\lambda}{(2z_2+a_2)\ldots\ldots(2z_r+a_r)} \quad \text{and} \quad \frac{\mu}{(2z_2+a_2)\ldots\ldots(2z_r+a_r)};$$

f_i, $(i=2, 3, \ldots\ldots, r)$, is a doubly periodic function of z_1 whose periods are

$$\frac{\lambda}{2(2z_2+a)\ldots\ldots(2z_{i-1}+a_{i-1})(z_i+a_i)(2z_{i+1}+a_{i+1})\ldots\ldots(2z_r+a_r)}$$

and

$$\frac{\mu}{2(2z_2+a_2)\ldots\ldots(2z_{i-1}+a_{i-1})(z_i+a_i)(2z_{i+1}+a_{i+1})\ldots\ldots(2z_r+a_r)};$$

f_0 and f_{ij}, $(i=r+1, \ldots\ldots, n;\ j=1, 2)$, are doubly periodic functions of z_1 whose periods are

$$\frac{\lambda}{2(2z_2+a_2)\ldots\ldots(2z_r+a_r)} \quad \text{and} \quad \frac{\mu}{2(2z_2+a_2)\ldots\ldots(2z_r+a_r)}.$$

Each period of any one of all f's except f_2 is in a commensurable ratio to the corresponding one of f_1 and those f's are therefore algebraic functions of f_1. Hence, in virtue of $(1)'$, f_2 must also be an algebraic function of f_1 and the relation (37) subsists among the periods of f_1 and of f_2. This is, however, impossible, since $\frac{2z_2+a_2}{z_2+a_2}$ is irrational. Consequently, we have $a_2=0$. Similarly $a_1=a_2=\ldots\ldots=a_r=0$.

As $f((z_i))$ is a doubly periodic function of z_j, $(j=1, 2, \ldots\ldots, r)$, we have

$$g_2^{(j)}(z_1,\ldots\ldots z_{j-1}z_{j+1}\ldots\ldots z_r)^4\bar{g}_2,\ g_3^{(j)}=(z_1\ldots\ldots z_{j-1}z_{j+1}\ldots\ldots z_r)^6\bar{g}_3,$$

and

$$\Delta^{(j)}=\{(g_2^{(j)})^3-27(g_3^{(j)})^2\}=(z_1\ldots\ldots z_{j-1}z_{j+1}\ldots\ldots z_r)^{12}(\bar{g}_2^3-27\bar{g}_3^2),$$

where $\bar{g}_2$ and $\bar{g}_3$ are absolute constants. If $f((z_i))$ be a simply periodic function of z_j, $\Delta^{(j)}\equiv 0$, or $\bar{g}_2^3-27\bar{g}_3^2=0$ and accordingly $f((z_i))$ is a simply periodic fuuction of $z_1z_2\ldots\ldots z_r$, i.e. it is an algebraic fnnction of $e^{Kz_1z_2\ldots\ldots z_r}$, where K is a certain constant different from zero.

23. We have hitherto discussed this problem under the assumption that at least one of ω_1 and ω_2 is dependent of z' and that at least one of Ω_1 and Ω_2 is dependent of z (§9). We now assume that at least one of ω_1 and ω_2 is dependent of z' while both Ω_1 and Ω_2 are independent of z. As Ω_1 and Ω_2 are independent z, we may conclude, from (23) and (26), that all $a(z')$'s are algebraic functions of $\wp(z',\ \Omega_1,\ \Omega_2,)$ and accordingly the same is true of $b(z')$'s, $(b_{mm_m}(z')\equiv 1)$, $g_2(z')$, $g_3(z')$ and $\zeta(z')$. Thus, $\zeta(z')$ is an algebraic function of $\wp(z',\ \Omega_1,\ \Omega_2)$, of $e^{Kz'}$, or of z', and in all cases, it must be a constant.[1] We have accordingly $\gamma(z')=\dfrac{1}{\alpha' z'+\beta'}$ (§§16–20). But $\gamma(z')=\sqrt[4]{\bar{g}_2/g_2(z')}$ is an algebraic function of $\wp(z',\ \Omega_1,\ \Omega_2)$ and we have consequently $\alpha'=0$, i.e., $\gamma(z')=$constant or $\Omega_1=\Omega_2=\infty$. If $\Omega_1=\Omega_2=\infty$, $f(z,\ z')$ and $b(z')$'s will be algebraic functions of z' for any assigned value of z, and so will

$$\wp(z,\ \omega_1(z'),\ \omega_2(z'))=(\alpha' z'+\beta')^2\wp(z(\alpha' z'+\beta'),\ \bar{\omega}_1,\ \bar{\omega}_2).$$

This is however impossible, and $\gamma(z')$ must be a constant, i.e., both $\omega_1(z')$ and $\omega_2(z')$ are independent of z'. Hence, *if one of the conditions*

i) *both* $\omega_1(z')$ *and* $\omega_2(z')$ *are independent of* z',

ii) *both* $\Omega_1(z)$ *and* $\Omega_2(z)$ *are independent of* z

subsists, then the other will follow necessarily.

24. From §§22 and 23, we may conclude that if $f((z_i))$ be an algebraic function of z_i, it can neither be an algebraic function of $\wp(z_j,\ \omega_1,\ \omega_2)$, $(i\gtrless j)$, nor an algebraic function of $e^{\alpha z_j}$, ω_1, ω_2, or α being dependent of z_i. For suitably chosen functions $\alpha'(z')$ and $\alpha(z)$, if $f(z,\ z')$ be an algebraic function of $\alpha'(z')(z+\alpha)$ and be, at the same time, an algebraic function of $\alpha(z)(z'+\alpha')$, in each of which the coefficients are

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[1] As $\zeta(z')$ acquires none of the value 0, 1 and ∞, it can not be an algebraic function of $\wp(z',\ \Omega_1,\ \Omega_2)$. By the same reasoning as in the proof that $K^2(z)$ is constant (§18), $\zeta(z')$ can not be an algebraic function of z' or of $x\equiv e^{Kz'}$.

independent of z and z', then $\alpha(z)$ and $\alpha'(z')$ must be algebraic functions of z and of z' respectively and $\alpha'(z')(z+a)$ is an algebraic function of $\alpha(z)(z'+a')$ whose coefficients are independent of z and z'.

$$Q_0(\alpha(z)(z'+a'))\{\alpha'(z')(z+a)\}^q+\cdots\cdots\cdots+Q_q(\alpha(z)(z'+a'))=0,$$

where $Q_0, \cdots\cdots\cdots, Q_q$ are polynomials of $\alpha(z)(z'+a')$ whose coefficients are independent of z and z'. If $\alpha'(z')$ has a zero point at finiteness, it must be a zero point of $Q_q(\alpha(z)(z'+a'))$ which depends on z unless it is $z'=-a'$. Similarly, if $\alpha'(z')$ has an infinity point at finiteness, it must also be the point $z'=-a'$. By the same reasoning, we may prove that if $\alpha'(z')$ has a branch point at finiteness, it must also be the point $z'=-a'$. Hence we may easily conclude that $\alpha'(z')\equiv A(z'+a')^{\frac{p'}{p}}$ where A is a constant different from zero and p, p' are integers, positive or negative; and accordingly $f(z, z')$ is an algebraic function of $(z+a)^p(z'+a')^{p'}$.

In general, $f((z_i))$ is an algebraic function of $(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\cdots\cdots\cdots\times(z_r+a_r)^{p_r}$ where $p_1, p_2, \cdots\cdots\cdots, p_r$ are integers, positive or negative, and $a_1, a_2, \cdots\cdots\cdots, a_r$ are constants.

Consequently $f((z_i))$ is an algebraic function of $(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\cdots\cdots\times(z_{r_1}+a_{r_1})^{p_{r_1}}, \cdots\cdots\cdots\cdots, (z_{r_{k-1}+1}+a_{r_{k-1}+1})^{p_{r_{k-1}+1}}\cdots\cdots\cdots\cdots(z_{r_k}+a_{r_k})^{p_{r_k}}, e^{\alpha z_{r_k+1}\cdots\cdots z_{r_k+1}}, \cdots\cdots\cdots, e^{\sigma z_{r_{s-1}+1}\cdots\cdots z_s}, \wp(z_{r_s+1}\cdots\cdots\cdots\cdots z_{r_{s+1}}, \omega_1^{(i)}, \omega_2^{(i)}), \cdots\cdots$

where p's are integers, positive or negative, and the coefficients, as well as $a_1, a_2, \cdots\cdots\cdots, a_{r_k}, \alpha, \cdots\cdots\cdots, \sigma, \omega_1^{(i)}, \omega_2^{(i)}, (i=1, 2, \cdots\cdots\cdots)$, are independent of the variables.

25. We now prove that if one of $a_1, a_2, \cdots\cdots\cdots, a_{r_1}$ be zero, then all the others will be also. For the sake of brevity, we first consider that $r_1=2$. Suppose that $a_1\neq 0$ while $a_2=0$, and eliminate z_{i1} and z_{i2}, $(i=1, 2, \cdots\cdots\cdots, n)$, from

$$f_{i1}\equiv f(z_{11}+z_{12}, \cdots\cdots\cdots, z_{i-11}+z_{i-12}, z_{i1}, z_{i+12}+z_{i+11}, \cdots\cdots\cdots, z_{n1}+z_{n2}),$$

$$f_{i2}\equiv f(z_{11}+z_{12}, \cdots\cdots\cdots, z_{i-11}+z_{i-12}, z_{i2}, z_{i+11}+z_{i+12}, \cdots\cdots\cdots, z_{n1}+z_{n2}),$$

and

$$z_{i1}+z_{i2}\equiv Z_i, \quad (i=1, 2, \cdots\cdots\cdots, n).$$

This may be done as in §4, and we have

$$(38)\quad \varphi_i(f_{i1}, f_{i2}, (Z_1+a_1)Z_2^{\frac{p_2}{p_2}}, Z_3, \ldots\ldots, Z_n)=0,\ (i=2, \ldots\ldots, n),$$

since f_{21}, f_{22} and f_{ij}. $(i=3, \ldots\ldots, n;\ j=1, 2)$, are algebraic functions of $(Z_1+a_1)z_{21}^{\frac{p_2}{p_1}}$, $(Z_1+a_1)z_{22}^{\frac{p_2}{p_1}}$ and $(Z_1+a_1)Z_2^{\frac{p_2}{p_1}}$ respectively. We have however for $i=1$

$$(39)\quad \varphi_1(f_{11}, f_{12}, (Z_1+2a_1)Z_2^{\frac{p_2}{p_1}}, Z_3, \ldots\ldots, Z_n)=0.$$

We may eliminate, as in §§5–7, $(Z_1+a_1)Z_2^{\frac{p_2}{p_1}}$, Z_3, ..., Z_n from (38) and $f_0\equiv f((Z_1+a_1)Z_2^{\frac{p_2}{p_1}}, Z_3, \ldots\ldots, Z_n)$, we have

$$(40)\quad \Psi_1(f_{21}, f_{22}, f_{31}, \ldots\ldots, f_{n2}, f_0)=0,$$

where Ψ_1 is an irreducible polynomial of the arguments and contains f_{21}, f_{22} explicitly.

Eliminating f_{21} from (1) and (40), we have

$$(41)\quad \Psi_2(f_{11}, f_{12}, f_{22}, f_{31}, \ldots\ldots, f_{n2}, f_0)=0,$$

which does not contain f_{22} explicitly. For otherwise, considering z_{22} as a variable and z_{11}, z_{12}, Z_2, z_{31}, ..., z_{n2} as constants, all f's in Ψ_2 except f_{22} are constants while f_{22} is a variable. This is however impossible by (41). As Ψ_2 is a function of f_{11}, f_{12}, f_{31}, ..., f_{n2}, f_0 and does not contain f_{21}, f_{22} and z's explicitly, (41) shows that we may eliminate Z_1, Z_2, ..., Z_n from $\varphi_i=0$, $(i=1, 3, \ldots\ldots, n)$, and $f_0\equiv f((Z_1+a_1)Z_2^{\frac{p_2}{p_1}}, Z_3, \ldots\ldots, Z_n)$. This is however impossible since φ_i, $(i=3, \ldots\ldots, n)$, are functions of $(n-1)$ independent variables $(Z_1+a_1)Z_2^{\frac{p_2}{p_1}}$, Z_3, ..., Z_n while φ_1 contains another variable Z_2 which is independent of the former ones. Accordingly if $a_2=0$, we have also $a_1=0$. We may similarly prove that if $f((z_i))$ be an algebraic function of $(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\ldots\ldots\ldots\ldots(z_{r_1}+a_{r_1})^{p_{r_1}}$, a_1, a_2, ..., a_{r_1} are or are not zero simultaneously. Similarly for a_{r_1+1}, a_{r_1+2}, ..., a_{r_2}; a_{r_2+1}, ..., a_{r_3}; .....

Consequently $f((z_i))$ *is an algebraic function of* $(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\ldots\ldots$

$$\times (z_{r_1}+a_{r_1})^{p_{r_1}} \cdots\cdots\cdots\cdots, \ (z_{r_{k-1}+1}+a_{r_{k-1}+1})^{p_{r_{k-1}+1}} \cdots\cdots\cdots\cdots (z_{r_k}+a_{r_k})^{p_{r_k}},$$

$$e^{\alpha z_{r_k+1} \cdots\cdots z_{r_{k+1}}}, \ \cdots\cdots\cdots, \ e^{\sigma z_{r_{s-1}+1} \cdots\cdots z_{r_s}}, \ \wp(z_{r_s+1} \cdots\cdots\cdots z_{r_{s+1}}, \omega_1^{(1)}, \omega_2^{(1)}), \ \cdots\cdots\cdots$$

where p's are integers, positive or negative, the coefficients as well as a_1, a_2, ..., a_{r_k}, α, ..., σ, $\omega_1^{(i)}$, $\omega_2^{(i)}$, $(i=1, 2, \cdots\cdots)$, *are independent of the variables and* a_{r_i+1}, a_{r_i+2}, ..., $a_{r_{i+1}}$, $(i=0, 1, \cdots\cdots, k-1$; $r_0=0)$, *are or are not zero simultaneously.*

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On Characteristic Equations of Analytic Functions of Many Independent Variables.

By

Ryô Yasuda.

(Received December 20, 1922.)

In my previous paper,[1] I investigated the property of an analytic function $f(z_1, z_2, \ldots\ldots, z_n)$ of n independent variables $z_1, z_2, \ldots\ldots, z_n$ which has an algebraic addition-theorem:

$$(1)\quad F(f_{11}, f_{12}, f_{21}, \ldots\ldots\ldots, f_{n2}, f_0)=0,$$

where F is an irreducible polynomial of $f_{11}, f_{12}, \ldots\ldots, f_{n2}, f_0$, containing all of them explicitly, and

$$f_{ij}\equiv f(z_{11}+z_{12}, \ldots\ldots, z_{i-11}+z_{i-12}, z_{ij}, z_{i+11}+z_{i+12}, \ldots\ldots, z_{n1}+z_{n2}),$$
$$(i=1, 2, \ldots\ldots, n;\ j=1, 2)$$
$$f_0\equiv f(z_{11}+z_{12}, \ldots\ldots, z_{n1}+z_{n2}).$$

We now define a characteristic equation of an analytic function $f(z_1, z_2, \ldots\ldots\ldots, z_n)$ of n independent variables $z_1, z_2, \ldots\ldots\ldots, z_n$ as follows:—

$$(2)\quad G(f_{11}, f_{12}, f_{21}, \ldots\ldots\ldots, f_{n2}, f_0)=0,$$

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[1] These memoirs, vol. VI, p. 251.

where G is an irreducible polynomial of $f_{11}, f_{12}, \ldots\ldots f_{u2}, f_0$, containing, all of them explicitly, and

$$f_{ij} \equiv f\left(\frac{z_{11}+z_{12}}{2}, \ldots\ldots, \frac{z_{i-11}+z_{i-12}}{2}, z_{ij}, \frac{z_{i+11}+z_{i+12}}{2}, \ldots\ldots, \frac{z_{n1}+z_{n2}}{2}\right),$$

$$(i = 1, 2, \ldots\ldots, u\,;\ j = 1, 2)$$

$$f_0 \equiv f\left(\frac{z_{11}+z_{12}}{2}, \ldots\ldots\ldots\ldots, \frac{z_{n1}+z_{n2}}{2}\right).$$

Then, in the case of *single* independent variable, *any analytic function which has an algebraic addition-theorem has also a characteristic equation, and conversely.*[1] The following is an outline of the proof given by M. Falk.

I. An analytic function which has an addition-theorem has also a characteristic equation.

Let

$$(3)\quad F(f(u), f(v), f(u+v)) = 0$$

be an algebraic addition-theorem of an analytic function $f(z)$. Then the equation (3) holds for certain branches of $f(z)$, i.e.

$$(4)\quad F(f_i(u), f_j(v), f_k(u+v)) = 0,$$

where f_i, f_j and f_k represent certain branches of f. By analytic continuations along suitably chosen paths in the u- and v- planes, we may obtain, from (4),

$$F(f_l(u'), f_l(v'), f_k(u'+v')) = 0,$$

where f_l is a branch of f. Put $u' = v' = \frac{u+v}{2}$. Then

$$(5)\quad F\left(f_l\left(\frac{u+v}{2}\right), f_l\left(\frac{u+v}{2}\right), f_k(u+v)\right) = 0.$$

[1] M. Falk: Ueber die Haupteigenschaften der-jenigen analytischen Funktionen eines Arguments, welche Additionstheoreme besitzen. (Nova Acta Regloe Societatis Scientiarum Upsaliensis. Ser. IV., Vol. 1, N. 8).

Eliminating $f_k(u+v)$ from (4) and (5), we have

$$(6)\quad G(f_i(u), f_j(v), f_l\Big(\frac{u+v}{2}\Big))=0,$$

where $G(f_i, f_j, f_l)$ is an irreducible polynomial of f_i, f_j and f_l. Thus we have a characteristic equation

$$G\Big(f(u), f(v), f\Big(\frac{u+v}{2}\Big)\Big)=0. \qquad \text{Q.E.D.}$$

II. An analytic function which has a characteristic equation has also an addition-theorem.

Let

$$(7)\quad G\Big(f(u), f(v), f\Big(\frac{u+v}{2}\Big)\Big)=0$$

be a characteristic equation of an analytic function $f(z)$.

(i) Suppose that $z=0$ be a regular point of $f(z)$. Let u, v, u', v' be any four values of z such that $u+v=u'+v'$. Then, by eliminating $f\Big(\frac{u+v}{2}\Big)\equiv f\Big(\frac{u'+v'}{2}\Big)$ from (7) and

$$(8)\quad G\Big(f(u'), f(v'), f\Big(\frac{u'+v'}{2}\Big)\Big)=0,$$

we have

$$H(f(u), f(v), f(u'), f(v'))=0,$$

where H is an irreducible polynomial of its arguments f's. As $z=0$ is a regular point, $\lim f(v')=b$ is finite and determinate, so that we may write down H in the form

$$\begin{aligned} H(f(u), f(v), f(u'), f(v'))\equiv & F(f(u), f(v), f(u')) \\ & +F_1(f(u), f(v), f(u')),\ (f(v')-b)+\cdots\cdots \\ & +F_r(f(u), f(v), f(u')),\ (f(v')-b)^r=0, \end{aligned}$$

where F's are polynomials of $f(u)$, $f(v)$, $f(u')$ and v' is a point in the vicinity of $v'=0$. As H is irreducible, $F(f(u), f(v), f(u')\not\equiv 0$. In the limit $v'=0$, we have $u'=u+v$ and accordingly

$$F(f(u), f(v), f(u+v))=0. \qquad \text{Q.E.D.}$$

(ii) Suppose that $z=0$ be a singular point of $f(z)$. It may be easily proved that there is a value $z=a$ of z such that $f(z)$ is regular in the vicinity of $z=a$ and $z=2a$.

Put $u=u'+a$, $v=v'+a$,

and

$$f(u)\equiv f(u'+a)\equiv g(u'), \quad f(v)\equiv g(v').$$

Then the new origin is a regular point of $g(u')$, and we have, by (7),

$$G\left(g(u'), g(v'), g\left(\frac{u'+v'}{2}\right)\right)=0,$$

since $\dfrac{(u'+a)+(v'+a)}{2}=\dfrac{u'+v'}{2}+a$. Accordingly, by (i),

$$F(g(u'), g(v'), g(u'+v'))=0,$$

or as $g(u')\equiv f(u'+a)\equiv f(u)$, $g(v')\equiv f(v)$, and $g(u'+v')\equiv f(u+v-a)$,

$$(8)\quad F(f(u), f(v), f(u+v-a))=0,$$

since $v=2a$ is a regular point of $f(v)$, by putting $v=2a$ and then by replacing u by $u+v-a$, we have

$$(10)\quad F(f(u+v-a), f(2a), f(u+v))=0,$$

where $f(u+v-a)$ will represent by analytic continuation as in §1, the same branch of f as that in (9) and accordingly they are identical. Eliminating $f(u+v-a)$ from (9) and (10), we have

$$F_1(f(u), f(v), f(u+v))=0. \qquad \text{Q.E.D.}$$

Now, what is the case of *many* independent variables? This is our present question, and the answer is as follows:—

Theorem. In order that an analytic function $f(z_1, z_2, \ldots\ldots, z_n)$ *of* n *independent variables* $z_1, z_2, \ldots\ldots, z_n$ *has a characteristic equation, it is necessary and sufficient that* $f(z_1, z_2, \ldots\ldots, z_n)$ *is an algebraic function of*

$$(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\ldots\ldots\ldots\ldots(z_{r_1}+a_{r_1})^{p_{r_1}},\ \ldots\ldots\ldots\ldots,$$

$$(z_{r_{s-1}+1}+a_{r_{s-1}+1})^{p_{r_{s-1}+1}}\ldots\ldots\ldots\ldots\ldots\ldots(z_{r_s}+a_{r_s})^{p_{r_s}},$$

$$e^{a_1(z_{r_s+1}+a_{r_s+1})\ldots\ldots\ldots\ldots(z_{r_{s+1}}+a_{r_{s+1}})}\ \ldots\ldots\ldots\ldots\ldots,$$

$$e^{a_{t-s}(z_{r_{t-1}+1}+a_{r_{t-1}+1})\ldots\ldots\ldots(z_{r_t}+a_{r_t})},$$

$$\wp((z_{r_t+1}+a_{r_t+1})\ldots\ldots\ldots\ldots(z_{r_{t+1}}+a_{r_{t+1}}),\quad \omega_1^{(1)},\ \omega_2^{(1)}),$$

$\ldots\ldots\ldots\ldots$, *where the coefficients aud* $a_1, a_2, \ldots\ldots\ldots\ldots, a_{r_{t+1}}, \ldots\ldots\ldots$, *are constants (zero being iucluded),* $p_1, p_2, \ldots\ldots\ldots, p_{r_s}$ *are integers positive or negative (zero being excluded), and* $a_1, \ldots\ldots\ldots, a_{t-s}, \omega_1^{(1)}, \omega_2^{(1)}, \ldots\ldots\ldots$ *are constants (zero being excluded and* $\frac{\omega_2^{(1)}}{\omega_1^{(1)}}, \ldots\ldots\ldots$ *being not real).*

As $\frac{(z_{i1}+a_i)+(z_{i2}+a_i)}{2}=\frac{z_{i1}+z_{i2}}{2}+a_i$, we may assume, without loss of generality, that all a's are zero and may thus prove the second part of the theorem analogously as in §§3–7 of my paper, *loc. cit.* For the first part of the theorem, the analogous proof to §§1–2, §§8–21 and §§23–24 of the same paper may hold; but since $\frac{(z_{i1}+a_i)+(z_{i2}+a_i)}{2}=\frac{z_{i1}+z_{i2}}{2}+a_i$, as before, it is unnecessary that all a's $e^{a_{i-s+1}(z_{r_i+1}+a_{r_i+1})\ldots\ldots\ldots\ldots(z_{r_{i+1}}+a_{r_{i+1}})}$, $(i=s, s+1, \ldots\ldots\ldots, t-1)$, and in $\wp((z_{r_i+1}+a_{r_i+1})\ldots\ldots\ldots(z_{r_{i+1}}+a_{r_{i+1}}),\ \omega_1^{(i-t+1)},\ \omega_2^{(i-t+1)})$, $(i=t, \ldots\ldots)$, are zero and that all a's in $(z_{r_i+1}+a_{r_i+1})^{p_{r_i+1}}\ldots\ldots\ldots\ldots\ldots(z_{r_{i+1}}+a_{r_{i+1}})^{p_{r_{i+1}}}$, $(i=0, 1, \ldots\ldots\ldots, s-1;\ r_0=0)$, are or are not zero simultaneously. Accordingly, *an analytic function of many independent variables which has an algebraic addition-theorem has also a characteristic equation, but the converse does not hold.*

On the Chemistry of Japanese Plants, I.

The Proximate Composition of Karafuto Wood.

By

Hisashi Nakamura.

(Received January 9, 1923.)

Yezo-matsu (*Picea ajanensis, Fisch.*) and Todo-matsu (*Abies sachalinensis, Masters.*) which are the principal constituents of the forests of Karafuto island, are regarded as the most important pulp material in Japan.

The cellulose-content of the wood was studied by I. Morioka,[1] from the point of view of paper manufacture. According to his report, the cellulose-content in Karafuto wood and European spruce is equal. The yield of sulphite pulp, however, which was made by the present author from our wood on a manufacturing scale at the Nippon Chemical Pulp Co., Karafuto, was small in comparison with that from European spruce, the yield being 10·5 and 9·7 tons of pulp from 22 tons of Yezo-matsu and Todo-matsu respectively, in the case of indirect cooking.

The present investigation, therefore, was undertaken in connection with the study of the manufacture of wood pulp, to the proximate composition of the wood.

The sample used in the experiment were sawdust from 130 years old trees grown at the Naibuchi, Karafuto, which are used in pulp manufacture, and were sifted with a 50-mesh sieve to get uniform size to avoid the effect of size of particle on yield of cellulose and other things.[2]

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1 Sapporo Norin Gakuho, **30** (1915); **40** (1920).

2 S. A. Mahood: J. Ind. & Eng. Chem., **12**, 833 (1920).

1. DRYING.

The sample was dried at 98° to constant weight, and the moisture was determined from the loss of weight.[1]

2. EXTRACTION WITH 95% ALCOHOL.

The etherial oil in the wood was studied by Shinozaki[2] and confirmed to consist mainly of pinene and phellandrene.

Wax, fat and resinous matter in the sample were estimated by extracting the dried substance with 95% alcohol boiling for three hours,[3] and the content calculated from the loss weight in Yezo-matsu and Todo-matsu is 3·1% and 3·8% respectively.

3. WATER EXTRACT.

Although cotton cellulose and filter paper, as obserbed independently by Robinoff[4] and Tauss,[5] are partially attacked by water on heating under pressure, pure cellulose is generally regarded by many investigators as insoluble in hot water.[6] However, pentosans and hexosans, according to S. Komatsu and his co-operators' experiments,[7] by heating with water under pressure were partially hydrolysed into simple sugars. It is, therefore, proper to ascribe the origin of simple sugars in the water extract of woods obtained under pressure to the hemicellulose which is composed of cell-wall.

Spruce heated with water under a pressure of five atmospheres for three hours, yields 1·18% reducing sugar, and the increased yield of the sugar in water extract results from increased pressure as seen in J. Konig and E. Rump's experiment.[8]

To determine the hydrolytic degree of the woods by water, 25 grm. of air-dried Yezomatsu, were heated with 300 c.c. water in autoclave for two hours, and the extract, separated from the insoluble matter by filtration, was diluted with water to 500 c.c. The reducing sugar in the

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1 M. Renker: U. Bestim. d. Cellulose, 1910, 10–20.

2 J. Chem. Ind. Japan., **15**, 730 (1912).

3 P. Klason: Beitrage z. Kent. d. Chem. Zusam. d. Fichtenholz, 1911, 34.

4 U. Einwirk. u. Wasser u. Natronlauge a. Baumwollcellulose, 1912, 21.

5 G. Schwalbe: Chem. d. Cellulose, 21–25.

6 G. Schwalbe: Chem. d. Cellulose, 1911, 19; H. Pringsheim: Die Polsach, 1919, 12

7 This memoirs, **5**, 308 (1922).

8 Chem. u. Struktur d. Pflanzen-zellmembran, 1914, 23; G. Schwalbe: Chem. d. Cell., 54, 396; P. Klason: Beitrag. z. Kennt d. Chem. Zusam. d. Fichtenholz, 1911, 29.

extract was estimated by means of the Fehling solution, and, as shown in Fig. 1., increased with temperature.

For comparison of the results obtained from Yezo-matsu and Todo-matsu, 5 grm. of each sample extracted with alcohol and dried, were digested with 70 c.c. water at 120–123° in a sealed tube for one hour, and the loss of weight of the sample and the reducing sugar-content in the water extract were found to be:

1·54 % loss of weight and 0·19 % sugar in Yezo-matsu.

1·65 % loss of weight and 0·29 % sugar in Todo-matsu.

Pentose in the water extract prepared from Yezo-matsu by heating with water at 145–148° for one hour, was determined by converting it into furfurol with hydrochloric acid by Tollens and Kruger's method.[1] It amounted to 39·8 % of the reducing sugars.

4. HYDROLYSIS WITH DILUTE HYDROCHLORIC ACID.

Of the polysaccharides occurring in plant cell-walls, only pentosans and hexosans of which hemi-cellulose is composed were hydrolysed by digestion with dilute hydrochloric and sulphuric acids, into simple sugars.[2] Such digestion, however, evidently had some effect upon the yield of sugars, cellulose being partially attacked by the acids. Klason[3] has reported that when spruce is digested with mineral acid at high temperature, it yields 22–23% of reducing sugar which consists of 25% xylose, 6% mannose, 64 % unknown sugar and a trace of galactose.

25 grm. of Yezo-mutsu sawdust were heated with 300 c.c. one per-cent hydrochloric acid for one hour in autoclave. The acidic solution was filtered and neutralised with sodium carbonate, and the reducing sugar content was determined by the Fehling solution. This reducing sugar content showed a gradual increase with temperature as shown in Fig. 1.

To compare the sugar content in the acid extracts obtained from both Yezo-matsu and Todo-matsu, 5 grm. of each sample were digested with 70 cc. 1 % hydrochloric acid in a sealed tube at 148–152° for one

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1 Tollens: Handbuch der Kohlenhydrat III Aufl. s. 137; Browne: Handbook of Sugar Analysis p. 459.

2 G. Schwalbe: p. 54; M. Kenker: p. 37; J. König and Rump: **23**, 73: Lunge: Chem. Tech. Unter. method., II, 453 (1904); E. Hägglund: D. Hydroly d. Zellulose u. d. Holzes, 430, 450; Kuhn: Zs. Physiol. Chem., **14**, 244 (1890).

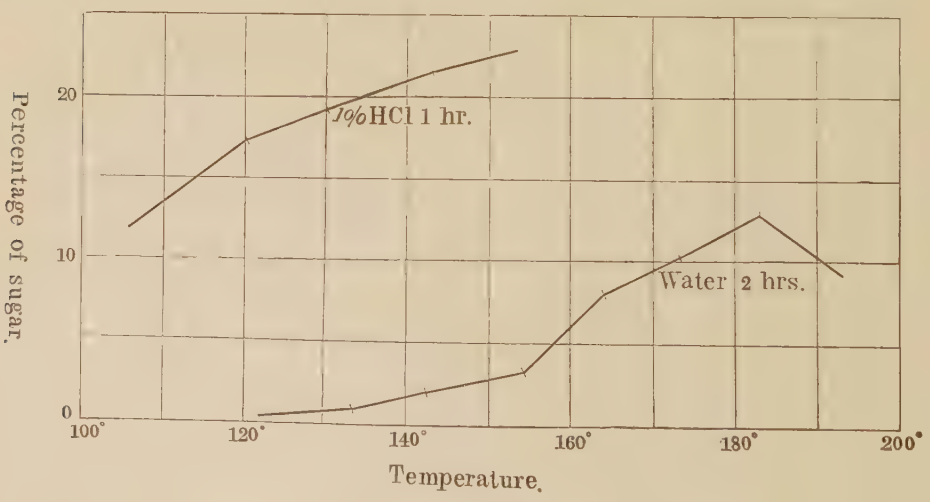
3 Papier Zeit., **35**, 2518 (1910); also compare Simonsen: Zs. angew. Chem., **111**, 195 (1898).

hour, and the following results were obtained:

37·7% (1·583 grm.) loss of weight and 21% sugar calculated as *d*-glucose in Yezomatsu.

54·9% (2·338 grm.) loss of weight and 22·8 % sugar in Todomatsu.

To estimate quantitatively the simple sugars in the acid extract obtained from both Todo-matsu and Yezo-matsu by boiling sawdust, after extracted with alcohol and dried, with 3% sulphuric acid for three hours, the acid extract was neutralized with barium carbonate to remove sulphuric acid, and then treated with basic lead acetate solution. The solution freed from lead salts by means of hydrogen sulphide was concentrated unner reduced pressure to small volume. One part of the solution was used for the estimation of pentoses by Tollens and Krugers's method.[1] The amount of mannose in the solution was determined by precipitating it as mannophenylhydrazone as usual. The amount of galactose was determined by oxydising another part of the solution with nitric acid into mucic acid, which was weighed after being separated from other sugar acids formed simultaneously from other hexoses associated with mannose in the solution, by repeated precipitation with hydrochloric acid from their sodium salts solution. The results were as follows:



[1] Loc cit.

	Yezo-matsu			Todo-matsu
	Water 145–148°, 1h.	1% HCl 146–152°, 1h.	3% H_2SO_4 boil 4h.	3% H_2SO_4 boil 4h.
Mannose	——	——	45·57%	43·04%
Galactose	——	——	9·67%	8·89%
Xylose	39·79%	16·03%	29·00%	24·59%
Pentose	48·87%		32·32%	27·04%

The proportion of the simple sugars in the acid extracts as seen from the above table, was quite different from that obtained by Klason.[1]

Referring to the investigation by E. Hägglund[2] on the compositions of the waste liquors from the sulphite process, the author's results seem to explain the real nature of the hemi-cellulose occuring in the wood. An experiment for alcoholic fermentation of the acid extract of the wood was undertaken with one main object to determine the fermentable sugar and consequently to ascertain whether the author's view on the composition of hemi-cellulose was correct.

The sawdust was boiled with 3% sulphuric acid for four hours, the acid extract was treated with basic lead acetate and the filtrate separated from lead salts was treated again with hydrogen sulphide to remove lead salts remaining in the solution. The filtrate from the precipitate of lead sulphide, amounting to 2000 c.c., contained 98 grm. of reducing sugar calculated as *d*-glucose. The solution, thus obtained, was found to ferment freely on adding bottom yeast obtained from the Nippon Brewery Co., Suita. The solution after being fermented and after the alcohol was distilled off, was treated with basic lead acetate. The quantity of sugar remaining in the solution without fermentation was estimated by means of the Fehling solution, to be equal to about 30% of the original sugar, amounting to 22 grm. of sugar.

5. PENTOSANS AND HEXOSANS.

It has already been mentioned that pentosan such as xylan occur in lignified cell-walls, either mixed or in combination with some hexosans such as mannan and galactan.

Pentosans in the samples were determined by Tollens and Kruger' method: the samples, extracted with alcohol and dried, were distilled

1 Loc. cit.

2 D. Sulfitablauge u. ihre Verarbeitung auf Alkohol, 1921, 15, and also compare G. Schwalbe: Chem. d. Cellulose, 419.

after digesting with hydrochloric acid, and furfurol in the distillate was precipitated with phloroglucin. Pentosans were calculated from the weight of furfurol phloroglucide as usual. The weight remaining after subtracting pentosans from the total reducing sugar, was designated as hexosans. The results are shown in table.

6. MANNOSE, GALACTOSE AND XYLOSE.

It was tried to isolate these sugars from the acid extract in the crystalline state, by which to give strong support to the theory of the existence of mannan, galactan and xylan in the wood cell-wall.

Sawdust from Yezo-matsu was hydrolysed with 3% sulphuric acid, and the acidic solution, separated from insoluble residue, was treated with basic lead acetate solution to remove sulphuric acid and lignin yielded by hydrolytic decomposition of ligno-cellulose. An excess of lead salt in the solution was removed completely by means of hydrogen sulphide, and the filtrate from the lead sulphide was then concentrated under reduced pressure after being decolourised with norite. Mannose in the solution was precipitated with phenylhydrazine hydrochloride and sodium acetate, and the hydrazone thus obtained was decomposed with benzaldehyde into mannose and benzalphenylhydrazone. The mannose solution freed from the hydrazone, was concentrated under reduced pressure to a thick syrup and an equal volume of hot glacial acetic acid was added. Standing the solution after seeding with *d*-mannose crystals, in an ice-box for a few days, *d*-mannose was crystallised, which was filtered from the mother liquor. To purify, the crystals were dissolved in water and concentrated under reduced pressure to a thick syrup, left in the ice-box to crystallise. The crystals were dried in the vacuo and their physical and chemical properties were studied, with the following results:

It melts at 132–133°, and having an initial rotatory power $[\alpha]_D = \frac{-18 \times 100}{2{\cdot}88} = -10°{\cdot}4$ (after 7 minutes) which on standing, becoming constant when $[\alpha]_D = +13°{\cdot}9$.

0·1489 grm. the substance gave 0·216 grm. CO_2, 0·0916 grm. H_2O and 0·0002 grm. ash.

	Calc. for $C_6H_{12}O_6$.	Found.
Carbon	40·00	39·88
Hydrogen	6·66	6.89
Ash	——	0·13

Another part of the solution which contained *d*-mannose, *d*-galactose *l*-xylose but no lignin, was oxidised with nitric acid, and mucic acid formed by the oxydation was isolated as usual from other sugar acids. It melts at 212°, and showed optically inactive in its boric acid solution.

0·1031 grm. of the acid gave 0·1287 grm. CO_2, 0·048 grm. H_2O.

	Calc. for $C_6H_{10}O_8$.	Found.
Carbon	34·29	34.06
Hydrogen	4·76	5·19

l-Xylose in the solution was confirmed by transforming into xylonic acid by oxydation with bromine, which was identified as the double salt of cadomium bromide with cadomium xylonate[1]

1). 0·2768 grm. sub. gave 0·1515 grm. $CdSO_4$.
2). 0·2504 grm. sub. gave 0·1276 grm. AgBr.

	Calc. for $(C_5H_9O_6)_2Cd + CdBr_2 + 2H_2O$.	Found.
Cd.	29·86	29·50
Br.	21·32	21·01

7. LIGNIN.

In old trees the cell-wall usually become lignified partially, and the presence of ligno-cellulose in wood affects the yield of pulp and also the consumption of the sulphite liquor in the course of the cooking of the wood.

Various methods have been proposed for the determination of lignin in wood, and the author used in this experiment the methods of J. Kŏnig,[2] Benedict and Bamberger.[3]

Sawdust extracted with alcohol and dried, was treated with 72% sulphuric acid at ordinary temperature according to the directions of Ost and Wilkennig,[4] where lignin only remains unaffected by the acid. Lignin insoluble in the acid was filtered in Gooch's crucible, washed with hot water and alcohol successively and then dried at 110° to constant weight. To confirm whether the substance insoluble in sulphuric acid, consisted of pure lignin, the methoxy group of the substance was estimated fol-

1 G. Bertrand, Bull. soc. chim., (3), **5**, 556 (1891)
2 Chem. u. struktur d. Pflanzen-Zell., 1914, 50.
3 Monat. Chem., **11**, 260 (1885).
4 Chem. Zeitung, **52**, 461 (1910).

lowing the Zeisel direction[1], using the apparatus modified by S. Zeisel and R. Fanto,[2] and the quantity of lignin was shown to be equal to 96% of the theoretical on the basis of the methoxy value.

0·1295 grm. of the sub. gave 0·299 grm. CO_2, 0·0572 grm. H_2O and 0·0029 grm. ash.

	Calc. for $C_{36}H_{40}O_{12}$.	$C_{38}H_{38}O_{12}$.	Found.
Carbon	65·06	66·47	64·40
Hydrogen	6·02	5·54	5·22
Methyl	5·88		5·08
Ash	——		2·24

Lignin content in the wood was also determined indirectly by estimating the methoxy value, following Benedikt and Bamberger's proposal,[3] and and the results were as shown in the table.

8. CELLULOSE.

Cellulose both in Yezo-matsu and Todo-matsu has already been determined by Morioka using Cross and Bevan's method. His results show fair agreement with the data obtained by the present author by the same method as seen in the table.

On the other hand, the author has treated the sawdust with dilute sulphuric acid and caustic potash solution successively, following Henneberg's directions,[4] and the cellulose content, thus obtained, is too large compared with others as shown in the table.

It is true that an absolutely accurate method of cellulose determiation as Renker[5] has already commented in his article, has not yet been devised. The method which has given the highest yield of pure cellulose, with ascertained minimum of change in the cellulose attending the process of isolation, so far as we can judge, is J. König's hydrogen peroxide method.[6]

To get a satisfactory result and to compare it with the other mentioned above, 3 grm. of air-dried sawdust after digesting with glycerol and sulphuric acid mixture, was treated with 120 cc. of 3% hydrogen peroxide and 10 c.c. of 24% ammonia solution.

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1 Monat. Chem., **6**, 989 (1885).

2 Zeitschrift. f. Anal. Chem., **42**, 549 (1903).

3 Monatsheft Chem. **15**, 509 (1894).

4 Lieb. Ann., **146**, 130 (1868).

5 U. Bestimmung. methoden d. Cellulose, **1910**, 86.

6 Ibid., 73.

After 12 hours, it was treated with 10 c.c. of 30% hydrogen peroxide, warmed in a water bath and then the residue was dried at 110° to constant weight. The substance thus obtained consisting of cellulose, cutin and ash, was ignited, and weighed. The differenee in the weighings corresponds with the amount of cellulose with some cutin. The percentage of cellulose and cutin in the dry sample was as shown in the table.

The content of nitrogenous matter determined by the Kjeldahl method and of ash in the sample, are shown in the table.

		Yezo-matsu.	Todo-matsu.
Water		15·50%	14·85%
Ash		0·78%	1·17%
Protein N.		0·59%	0·55%
Alcohol soluble matter		3·10%	3·78%
Pentosans		9·46%	9·57%
Polysoccharides (cal. as *d*-glucose)		21·79%	22·83%
Cellulose, by	Henneberg's method	71·94%	68·90%
	Cross & Bevan's m.	56·97%	56·41%
	Konig's m.	52·30%	47·52%
Lignin, by	72% H_2SO_3	35·23%	34·15%
	Methoxy value	43·52%	46·17%

	Materials.	temp.	hour.	loss. %	sugar %
Yezo-matsu	5 grm. 60 c.c. H_2O	120°	1	1·54	0·19
	5 grm. 70 c.c. 1% HCl	148–151°	1	37·7	21·79
	4 grm. 40 c.c. 72% H_2SO_4	ord. temp.	1	64·77	44·20
Todo-matsu	5 grm. 60 c.c. H_2O	120°	1	1·65	0·29
	5 grm. 70 c.c. 1% HCl	148–151°	1	54·92	22·83
	4 grm. 40 c.c. 72% H_2SO_4	ord. temp.	1	65·85	44·95

9. SUMMARY.

The content of moisture, ash, nitrogenous matter, resinous matter, lignin, pentosans, hexosans and cellulose both in Yezo-matsu and Todo-matsu was determined.

2. Todo-matsu contains much more pentosan, hexosan and less cellulose than Yezo-matsu.

3. *d*-Mannose was isolated from the dilute mineral acid extract in

a crystalline state, and *d*-galactose and *l*-xylose were confirmed to exist in the extract by converting into mucic and xylonic acids respectively.

The author takes this opportunity to express his thanks to Mr. H. Ueda for his invaluable assistance to carry out this investigation.

October 1922,
Laboratory of Organic and Biochemistry.

The Reaction between α-Pinene and Acids.
1. α-Pinene and Sulphuric acid.

By

Kashichi Ono.

(Received February 19, 1923.)

As is well known α-pinene gives terpinhydrate[1] by the catalytic action of dil. sulphuric acid. Recently O. Aschan[2] reported to have thus obtained terpinhydrate as the main hydration product (about 80% yield) with a small quantity of transterpine. The details of this experiment are, however, not given in the abstracts, and much seems not to have been studied about the influence of external conditions upon the reaction. So I have taken up the problem to study it more fully and, if possible, to clear up the true nature of the catalytic hydration.

EXPERIMENTAL PART.

α-Pinene obtained by repeated distillation of American turpentine oil having the following physical properties was used in the experiment.

Boiling point	155–156°
Specific gravity	0·8544 (24°)
Optical rotation	+37·81
Refractive index	1·4682 (21°)

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1 F. W. Semmler, Die ätherischen Oele, Bd. II., 203 (1906).

2 Journ. chem. Soc., Abstracts i, 336 (1919).

1. PRELIMINARY EXPERIMENT.

100 grams of α-pinene were mixed with 300 grams of sulphuric acid (about 43%), and the mixture was vigorously shaken for about 15 hours at 18–20°C. After separating the crystalline substance from the reaction product by filtration, the filtrate was neutralized with sodium carbonate. The crystalline substance first separated and solid matter isolated by neutralization were mixed together and washed several times with petroleum ether. The crystalline mass thus purified, after having been dried in air, was crystallized from alcohol, whereupon needles, melting at 116–117° were obtained. The yield was about 31·5 grams. The crystals were proved to be terpinhydrate by a direct comparison with a pure specimen.

The oily liquid portion (about 66 grams) obtained after neutralization of sulphuric acid by soda and separated from terpinhydrate gave the following physical characters:

Specific gravity	0·8737 (22°)
Optical rotation	+5·25
Refractive index	1·4773 (21°)

50 c.c. of the oil were subjected to distillation under atmospheric pressure, and the following fractions were collected:

155–160°	3·1 c.c.
160–165°	1·8 c.c.
165–180°	39·9 c.c.
Above 180°	4·8 c.c.

The principal distillate (B.P. 165–180°) was found to have the following physical properties.

Specific gravity	0·8685 (17°)
Optical rotation	almost inactive
Refractive index	1·4751 (21°)

This fraction was redistilled on metallic sodium under atmospheric pressure, and a fraction boiling at 175–178° was collected and examined.

Specific gravity	0·8572 (17°)
Optical rotation	±0
Refractive index	1·4745 (21°)

It was analysed with the following results:
0·1123 gr. substance gave 0·3628. gr. CO_2 and 0·1209 gr. H_2O

	Found	Cal. ($C_{10}H_{16}$)
Carbon	88·09%	88·15%
Hydrogen	12·03%	11·85%

A solution of this fraction (5·4 c.c.) in three times its volume of a mixture consisting of 2 parts of amyl alcohol and 3 parts of ether was cooled with a freezing mixture, and into it a solution of bromine (2·8 c.c.) in an amyl-alcohol-ether mixture was added gradually. 2 or 3 drops of water were then added, and the reaction mixture was allowed to stand overnight in an open beaker, when a large crop of white crystals was produced. On recrystallizing them from petroleum ether crystals melting at 124° and having all the properties of dipentenetetrabromide were obtained.

2. QUANTITATIVE INVESTIGATION.

To learn the influence of various conditions upon the reaction following experiments were conducted:

(1) Influence of the concentration of sulphuric acid.

α-Pinene	100 grs.
Sulphuric acid	300 grs.
Time	5 hours.
Temperature	17–20°

Table I.

Concentration of sulphuric acid	Terpinhydrate	Oily part				Distillation test (50 c.c.)			
		gr.	sp. gr.	optical rotation	refractive index	155–160°	160–165°	165–178°	above 178°
%	gr.		(24°)		(22°)	c.c.	c.c.	c.c.	c.c.
10	no formation	99·3	0·8584	+31·80	1·4693	47·1	2·6 (above 160°)		
20	trace	99·2	0·8582	+31·03	1·4694	40·5	7·4	1·6 (above 165°)	
30	1·4	99·0	0·8610	+21·37	1·4696	33·4	11·9	3·8 (above 165°)	
40	13·2	87·7	0·8624	+15·98	1·4697	28·3	13·2	6·8	0·9
45	32.0	68·3	0·8727	+14·35	1·4700	9·4	13·8	21·0	4·7
50	17·5	82·6	0·8626	+12·75	1·4728	7·8	7·7	28·5	4·9

(2) Influence of amount of sulphuric acid.

α-Pimene . . . . . 100 grs.
Time . . . . . 5 hours.
Temperature . . . . . 17–20°

Table II.

α-Pinene	Sulphuric acid (45%)	Terpinhydrate	Oily part				Distillation test (50 c.c.)			
			gr.	sp. gr.	optical relation	refractive index	155–160°	160–165°	165–178°	above 178°
gr.	gr.	gr.		(24°)		(22°)	c.c.	c.c.	c.c.	c.c.
100	100	19·8	81·4	0·8700	+14·83	1·4687	26·2	18·7	3·0	1·8
100	200	29·5	70·9	0·8737	+14·90	1·4690	9·7	15·8	18·9	3·3
100	300	32·0	68·3	0·8727	+14·35	1·4700	9·4	13·8	21·0	4·7
100	400	13·2	88·6	0·8759	+10·72	1·4738	8·0	11·9	25·3	3·2
100	500	10·7	91·7	0·8672	+ 8·64	1·4743	5·9	8·4	28·6	5·9

(3) Influence of time of reaction.

α-Pinene . . . . . 100 grs.
Sulphuric acid (45%) . . . . . 300 grs.
Temperature . . . . . 17–20°

Table III.

Time	Terpinhydrate	Time / Terpinhydrate	Oily part				Distillation test (50 c.c.)			
			gr.	sp. gr.	optical rotation	refractive index	155–160°	160–165°	165–178°	above 178°
hrs.	gr.	gr./hr.		(24°)		(22°)	c.c.	c.c.	c.c.	c.c.
5	32·0	6·4	78·3	0·8727	+14·35	1·4700	9·4	13·8	21·0	4·7
10	48·0	4·8	52·2	(29°) 0·8780	+10·45	1·4708	1·2	9·4	28·9	8·8
15	53·5	3·5	46·4	(29°) 0·8739	+ 5·27	1·4773	*0·7	5·7	26·0	7·0

* Distillation test for 40 c.c.

(4) Influence of temperature.

α-Pinene . . . . . 100 grs.
Sulphuric acid (45%) . . . . . 300 grs.
Time . . . . . 10 hours.

Table IV.

Temperature	Terpin hydrate	Oily Part				Distillation test (50 c.c.)			
		gr.	sp. gr.	optical rotation	refractive index	155–160°	160–165°	165–178°	above 178°
	gr.		(29°)		(22°)	c.c.	c.c.	c.c.	c.c.
17–20°	48·4	52·2	0·8780	+10·45	1·4708	1·2	9·4	28·9	8·8
0–(–6°)	82·5	17·7	0·8640	+17·00	1·4683				
0–(–4°)	80·1	20·8	0·8645	+17·38	1·4680	28·5	7·5	19·0	3·2
0–(–7°)	83·0	17·9	0·8640	+17·16	1·4684				

3. ACTION OF SULPHURIC ACID UPON TERPINEOL.

The terpineol used in the experiments was prepared in the manner described below.

200 grams of terpinhydrate were mixed with 350 grams of oxalic acid (about 0·5%), and the mixture was boiled for 5 hours in an oil bath under constant stirring. The upper layer, coloured pale brown, was separated from the oxalic acid solution, washed with a dil. soda solution and then with water and subjected to steam distillation. The oily distillate dried with anhydrous sodium sulphate, was distilled under diminished pressure. The yield was about 85%.

Boiling point	103–107° (12–14 m.m.)
Specific gravity	0·9356 (20°)
Refractive index	1·4822 (21°)

50 grams of terpineol thus prepared were mixed with 200 grams of sulphuric acid (45%), and the mixture, cooled with a freezing mixture, was vigorously shaken for about 8 hours. The crystals obtained, were washed several times with petroleum ether and recrystallized from alcohol. The yield was almost quantitative. The crystals melted at 116–117° and were proved to be identical with terpinhydrate.

4. ACTION OF SULPHURIC ACID UPON DIPENTENE.

Dipentene, showing the following physical properties, was used.

Specific gravity	0·8563 (16°)
Optical rotation	±0
Refractive index	1·4738 (21°)

50 grams of dipentene were treated with 200 grams of 45% sulphuric acid in the manner analogous to that described in the previous article, and crystals proved to be terpinhydrate were obtained.

5. PREPARATION OF TERPINHYDRATE FROM TURPENTINE OIL.

Turpentine oil was distilled on metallic sodium, and a fraction boiling at 155–160° was collected. 100 grams of this fraction were mixed with 300 grams of sulphuric acid (45%), and the mixture cooled with a freezing mixture was vigorously shaken for 10–15° hours.

The crystals separated and those obtained by neutralizing the filtrate with crystalline sodium carbonate were mixed together and washed several times with petroleum ether. The white crystals when recrystallized from alcohol, melted at 116–117°. The yield was about 80 grams.

6. MECHANISM OF REACTION.

It is already known that α-pinene changes into terpineol[1] on the one hand and into transterpine[2] on the other, by the action of sulphuric acid. Now it might be better to regard these transformations and those described above as proceeding continuously from one to another than to consider them all independent. From this point of view I propose the following scheme for the catalytic hydration of α-pinene which takes place through the agency of sulphuric acid:

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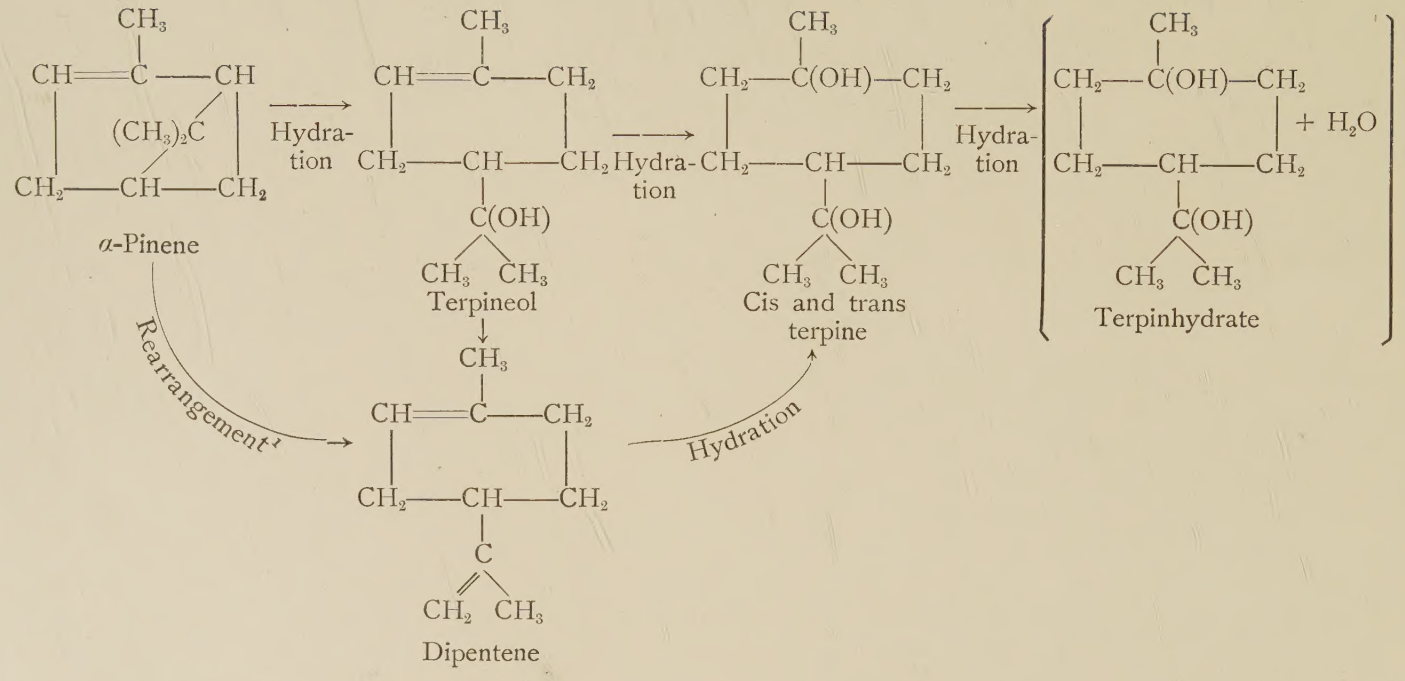
1 Flavitzky, Ber. D. chem. Ges., **12**, 2354 (1879).

2 O. Aschan, loc. cit.



F. W. Semmler, Die ätherischen Oele, Bd. II., 215 (1906).

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